Power: twenty-three questions with solutions

Power of a steady force, of a changing force and of gravity; average power; motion at constant power; pumps, water jets and a windmill; friction and air resistance. Every number in a solution is worked out by ApasScript. Where it is needed, take the acceleration of free fall as 10 $\mathrm{m/s^2}$ and the density of water as 1000 $\mathrm{kg/m^3}$.

Question 1. A steady push: power now and power on average

A trolley of mass 4 kg stands on a smooth level floor. From a certain moment a steady horizontal force of 12 N pushes it. Find (a) the power delivered by the force 5 s after the start, and (b) the average power over those 5 s.

Solution

Power is the rate at which a force does work. For a force along the motion it is force times speed:

$$P = \frac{dW}{dt} = F\,v$$

(a) The force is steady, so the acceleration is steady and the speed grows in proportion to time:

$$\begin{aligned} a &= \frac{F}{m} \\ &= \frac{12}{4} \\ &= 3\,\mathrm{m/s^2} \\ v &= aT \\ &= 3 \times 5 \\ &= 15\,\mathrm{m/s} \\ P &= Fv \\ &= 12 \times 15 \\ &= 180\,\mathrm{W} \end{aligned}$$

(b) Average power is the total work divided by the time taken. The trolley starts from rest:

$$
\begin{aligned}
s &= \frac{aT^2}{2} \\
&= \frac{3 \times 5^2}{2} \\
&= \frac{75}{2}\,\mathrm{m} \\
W &= Fs \\
&= 12 \times \frac{75}{2} \\
&= 450\,\mathrm{J} \\
A &= \frac{W}{T} \\
&= \frac{450}{5} \\
&= 90\,\mathrm{W}
\end{aligned}
$$

> Result.
>
> After 5 s the force delivers 180 W. Over the 5 s it delivers 90 W on average: half the final value, because the speed, and with it the power, rises steadily from zero.

Question 2. Force and velocity in different directions

A drone flies in a horizontal plane with velocity components 3 m/s along the x axis and 4 m/s along the y axis. At that moment the wind pushes on it with a force of components 6 N and -2 N. Find the power delivered to the drone by the wind, and the angle between the force and the velocity.

Solution

Only the part of the force along the velocity does work. In a short time dt the drone moves by $v\,dt$, so the work is $F \cdot v\,dt$ and the power is the dot product:

$$P = F \cdot v = F_x\,v_x + F_y\,v_y$$

$$
\begin{aligned}
P &= F_x v_x + F_y v_y \\
&= 6 \times 3 + -2 \times 4 \\
&= 10\,\mathrm{W}
\end{aligned}
$$

The angle follows from $\cos\theta = P/(F\,v)$:

$$
\begin{aligned}
F &= \sqrt{F_x^2 + F_y^2} \\
&= \sqrt{6^2 + (-2)^2} \\
&= 2\sqrt{10}\,\mathrm{N} \\
v &= \sqrt{v_x^2 + v_y^2} \\
&= \sqrt{3^2 + 4^2} \\
&= 5\,\mathrm{m/s} \\
c &= \frac{P}{Fv} \\
&= \frac{10}{2\sqrt{10} \times 5} \\
&= \frac{\sqrt{10}}{10}
\end{aligned}
$$

> Result.
>
> The wind delivers 10 W. The cosine of the angle is $\frac{\sqrt{10}}{10}$, so the force is at about $72°$ to the velocity. A force at right angles to the velocity would deliver no power at all.

Question 3. A truck climbing at steady speed

A truck of mass 1200 kg climbs a straight road that rises 1 m for every 20 m travelled along it. The resistance to its motion is 400 N. Find the power its engine must deliver to the wheels to keep a steady speed of 15 m/s.

Solution

At steady speed the forces along the road balance. The driving force equals the part of the weight along the slope plus the resistance. The slope has $\sin\alpha$ equal to $1/20$:

$$
\begin{aligned}
F &= m\,g\,\sin\alpha + R \\
G &= \frac{mg}{n} \\
&= \frac{1200 \times 10}{20} \\
&= 600\,\mathrm{N} \\
F &= G + R \\
&= 600 + 400 \\
&= 1000\,\mathrm{N} \\
P &= Fv \\
&= 1000 \times 15 \\
&= 15000\,\mathrm{W}
\end{aligned}
$$

> Result.
>
> The engine must deliver 15000 W, that is 15 kW.

Question 4. A force that grows with time

A block of mass 2 kg rests on a smooth level table. From the moment $t = 0$ a horizontal force acts on it that grows in proportion to time:

$$F = b\,t$$

with b equal to 6 N/s. Find (a) the power delivered by the force at $t = 4$ s, and (b) the average power over the first 4 s.

Solution

(a) The acceleration is $F/m = b\,t/m$. It is not steady, so the speed is found by adding it up over time:

$$v = \int_0^t \frac{b\,t}{m}\,dt = \frac{b\,t^2}{2\,m}$$

The power is the force at that moment times the speed at that moment:

$$P = F\,v = \frac{b^2\,t^3}{2\,m}$$

$$\begin{aligned} F &= bT \\ &= 6 \times 4 \\ &= 24\,\mathrm{N} \\ v &= \frac{bT^2}{2m} \\ &= \frac{6 \times 4^2}{2 \times 2} \\ &= 24\,\mathrm{m/s} \\ P &= Fv \\ &= 24 \times 24 \\ &= 576\,\mathrm{W} \end{aligned}$$

(b) The power changes with time, so the work is the area under the power-time curve:

$$W = \int_0^T \frac{b^2\,t^3}{2\,m}\,dt = \frac{b^2\,T^4}{8\,m}$$

$$
\begin{aligned}
W &= \frac{b^2 T^4}{8m} \\
&= \frac{6^2 \times 4^4}{8 \times 2} \\
&= 576 \,\mathrm{J} \\
A &= \frac{W}{T} \\
&= \frac{576}{4} \\
&= 144 \,\mathrm{W}
\end{aligned}
$$

> Result.
>
> At 4 s the force delivers 576 W. The average over the first 4 s is 144 W, a quarter of the final value: the power grows as the cube of the time, so most of the work is done near the end.

Question 5. A thrust that falls off with distance

A sledge of mass 2 kg starts from rest at $x = 0$ on smooth ice. A small rocket pushes it along the x axis with a force that weakens as the sledge moves:

$$F = F_0 - k\,x$$

with $F_0 = 20$ N and k equal to 2 N/m. Find the power delivered by the rocket when the sledge passes $x = 4$ m.

Solution

Power needs the force and the speed at that point. The force is read from the law. The speed comes from the work done so far, which is the area under the force-distance curve:

$$W = \int_0^{x} (F_0 - k\,x)\,dx = F_0\,x - \tfrac{1}{2}\,k\,x^2$$

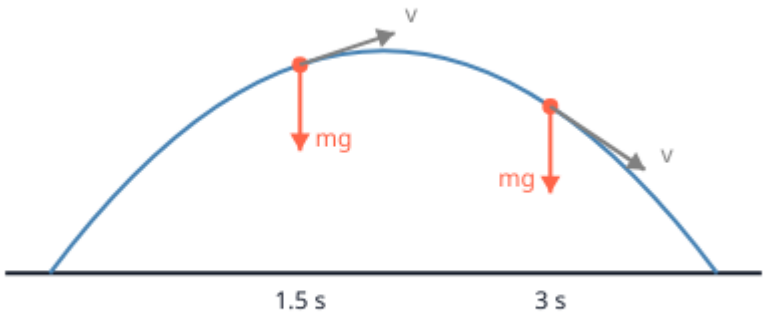
The sledge started from rest, so this work is its kinetic energy, $\tfrac{1}{2}\,m\,v^2 = W$:

$$
\begin{aligned}
W &= F_0 x_1 - \frac{k x_1^2}{2} \\
&= 20 \times 4 - \frac{2 \times 4^2}{2} \\
&= 64\ \text{J} \\
v &= \sqrt{\frac{2W}{m}} \\
&= \sqrt{\frac{2 \times 64}{2}} \\
&= 8\ \text{m/s} \\
F &= F_0 - k x_1 \\
&= 20 - 2 \times 4 \\
&= 12\ \text{N} \\
P &= F v \\
&= 12 \times 8 \\
&= 96\ \text{W}
\end{aligned}
$$

> Result.
>
> At 4 m the rocket delivers 96 W. The force has dropped to 12 N by then, but the sledge is moving at 8 m/s.

Question 6. Power of gravity on a thrown ball

A ball of mass 0.5 kg is thrown from level ground with a horizontal velocity of 15 m/s and an upward velocity of 20 m/s. Air resistance is negligible. Find the power of the force of gravity (a) 1.5 s after the throw, (b) 3 s after the throw, and (c) on average over the whole flight.



Solution

Gravity acts straight down with the force $m\,g$. Only the vertical part of the velocity counts in the dot product. Taking up as positive, the vertical velocity is $u_y - g\,t$, so

$$P = F \cdot v = -\,m\,g\,(u_y - g\,t)$$

The horizontal velocity does not enter at all.

$$\begin{aligned} w_1 &= u_y - gT_1 \\ &= 20 - 10 \times 1.5 \\ &= 5\,\mathrm{m/s} \\ P_1 &= -mgw_1 \\ &= -0.5 \times 10 \times 5 \\ &= -25\,\mathrm{W} \\ w_2 &= u_y - gT_2 \\ &= 20 - 10 \times 3 \\ &= -10\,\mathrm{m/s} \\ P_2 &= -mgw_2 \\ &= -0.5 \times 10 \times (-10) \\ &= 50\,\mathrm{W} \end{aligned}$$

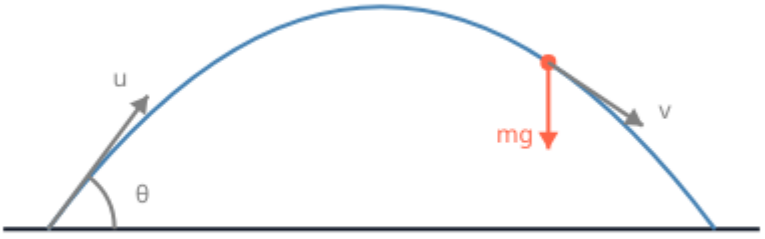
(c) The ball lands at the height it left, so over the whole flight gravity does no work in total, and the average power is zero. The time of flight, the height at landing and the direct average confirm it:

$$\begin{aligned} T_f &= \frac{2u_y}{g} \\ &= \frac{2 \times 20}{10} \\ &= 4\,\mathrm{s} \\ y_f &= u_y T_f - \frac{gT_f^2}{2} \\ &= 20 \times 4 - \frac{10 \times 4^2}{2} \\ &= 0\,\mathrm{m} \\ W &= -mgy_f \\ &= -0.5 \times 10 \times 0 \\ &= 0\,\mathrm{J} \\ A &= \frac{W}{T_f} \\ &= \frac{0}{4} \\ &= 0\,\mathrm{W} \end{aligned}$$

> Result.
>
> At 1.5 s the power of gravity is -25 W: the ball is still rising and gravity takes energy from it. At 3 s it is 50 W: the ball is falling and gravity gives energy back. Over the whole flight of 4 s the average is 0 W.

Question 7. Power of gravity at time t, for any throw

A stone of mass m is thrown from level ground with speed u at an angle θ above the horizontal. Air resistance is negligible. Find the power of the force of gravity at a time t after the throw. When is it zero? Find also its average value over the first t seconds.



Solution

Take the x axis horizontal and the y axis upward. Gravity has no horizontal part, and the horizontal velocity never changes; the vertical velocity falls steadily:

$$F = -m\,g\,\hat{j}$$

$$v = u\cos\theta\,\hat{i} + (u\sin\theta - g\,t)\,\hat{j}$$

Power is the dot product of force and velocity. Only the y parts multiply:

$$P = F\cdot v = -m\,g\,(u\sin\theta - g\,t)$$

The power is zero when the bracket is zero, that is, when the vertical velocity is zero: at the top of the path, at the time

$$t_{\text{top}} = \frac{u\sin\theta}{g}$$

Before that the power is negative (the stone is rising and gravity takes energy from it); after that it is positive. It changes with time along a straight line, starting from $-m\,g\,u\sin\theta$.

For the average, divide the work of gravity by the time. The work is $-m\,g$ times the height reached, and the height is $u\sin\theta\,t - \frac{1}{2}\,g\,t^2$:

$$P_{\text{av}} = \frac{-m\,g\,\left(u\sin\theta\,t - \frac{1}{2}\,g\,t^2\right)}{t} = -m\,g\,\left(u\sin\theta - \tfrac{1}{2}\,g\,t\right)$$

> Result.
>
> $$P = m\,g\,(g\,t - u\sin\theta)$$
>
> It is zero at the top of the path. Its average over the first t seconds is $m\,g\,(\frac{1}{2}\,g\,t - u\sin\theta)$, which is zero over the whole flight, $t = 2\,u\sin\theta/g$.

Question 8. Gravity on the way up only

For the ball of Question 6 (mass 0.5 kg, upward velocity 20 m/s at the throw), find the average power of gravity over the rising part of the flight only.

Solution

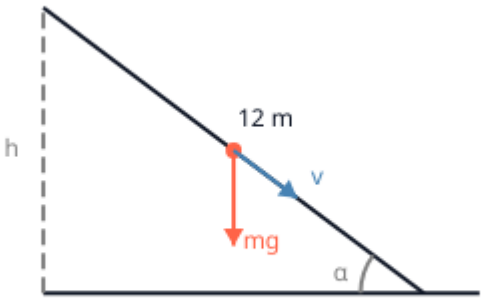
The ball rises until its vertical velocity is zero. During the rise gravity does the work $-m\,g\,h$, where h is the greatest height:

$$\begin{aligned} T_u &= \frac{u_y}{g} \\ &= \frac{20}{10} \\ &= 2\,\mathrm{s} \\ h &= \frac{u_y^2}{2g} \\ &= \frac{20^2}{2 \times 10} \\ &= 20\,\mathrm{m} \\ W &= -mgh \\ &= -0.5 \times 10 \times 20 \\ &= -100\,\mathrm{J} \\ A &= \frac{W}{T_u} \\ &= \frac{-100}{2} \\ &= -50\,\mathrm{W} \end{aligned}$$

> Result.
>
> Over the rise gravity delivers -50 W on average. Over the fall it delivers the same amount with a plus sign, which is why the average over the whole flight is zero.

Question 9. A block sliding down a smooth ramp

A block of mass 2 kg is released from rest at the top of a smooth ramp 12 m long. The ramp drops 3 m for every 5 m along it. Find the power of gravity at the moment the block reaches the bottom, and the average power of gravity during the slide.



Solution

The ramp has $\sin\alpha$ equal to $3/5$. Along the ramp the only force with a part in that direction is gravity, $m\,g\,\sin\alpha$, so the acceleration is $g\,\sin\alpha$:

$$\begin{aligned} a &= \frac{gp}{q} \\ &= \frac{10 \times 3}{5} \\ &= 6\,\mathrm{m/s^2} \\ v &= \sqrt{2aL} \\ &= \sqrt{2 \times 6 \times 12} \\ &= 12\,\mathrm{m/s} \end{aligned}$$

The velocity points down the ramp and gravity points straight down. The angle between them is $90^\circ - \alpha$, so the power of gravity is

$$P = m\,g\,v\,\cos(90^\circ - \alpha) = m\,g\,v\,\sin\alpha$$

$$\begin{aligned} P &= \frac{mgvp}{q} \\ &= \frac{2 \times 10 \times 12 \times 3}{5} \\ &= 144\,\mathrm{W} \end{aligned}$$

For the average, divide the work of gravity by the time of the slide. The block drops through the height h of the ramp:

$$
\begin{aligned}
h &= \frac{Lp}{q} \\
&= \frac{12 \times 3}{5} \\
&= 7.2\,\mathrm{m} \\
W &= mgh \\
&= 2 \times 10 \times 7.2 \\
&= 144\,\mathrm{J} \\
T &= \frac{v}{a} \\
&= \frac{12}{6} \\
&= 2\,\mathrm{s} \\
A &= \frac{W}{T} \\
&= \frac{144}{2} \\
&= 72\,\mathrm{W}
\end{aligned}
$$

> Result.
>
> At the bottom gravity delivers 144 W. Over the slide it delivers 72 W on average: half as much, because the block speeds up steadily from rest.

Question 10. A hoist: average power and greatest power

A hoist lifts a load of mass 800 kg from rest with a steady upward acceleration of $2\ \mathrm{m/s^2}$ for 4 s. Find the average power delivered by the cable over this time and the greatest power the motor has to give.

Solution

The cable must both hold the load and speed it up. From Newton's second law for the load, $F - m\,g = m\,a$:

$$
\begin{aligned}
F &= m(g + a) \\
&= 800 \times (10 + 2) \\
&= 9600\,\mathrm{N} \\
h &= \frac{aT^2}{2} \\
&= \frac{2 \times 4^2}{2} \\
&= 16\,\mathrm{m} \\
W &= Fh \\
&= 9600 \times 16 \\
&= 153600\,\mathrm{J} \\
A &= \frac{W}{T} \\
&= \frac{153600}{4} \\
&= 38400\,\mathrm{W}
\end{aligned}
$$

The cable's force is steady and the speed is greatest at the end, so the power is greatest at the end:

$$
\begin{aligned}
v &= aT \\
&= 2 \times 4 \\
&= 8\,\mathrm{m/s} \\
P &= Fv \\
&= 9600 \times 8 \\
&= 76800\,\mathrm{W}
\end{aligned}
$$

> Result.
>
> The cable delivers 38.4 kW on average and 76.8 kW at the end. A motor chosen by the average alone would be too weak by a factor of two.

Question 11. Constant power: speed and distance against time

A toy car of mass 2 kg starts from rest on a level track. Its motor delivers a constant power of 16 W to the car, and resistance is negligible. Find its speed and the distance it has covered 4 s after the start.

Solution

Write the constant power as N. Constant power means the work done grows in proportion to time, $W = N\,t$. All of it becomes kinetic energy:

$$
\tfrac{1}{2}\,m\,v^2 = N\,t
$$

$$
v = \sqrt{\frac{2\,N\,t}{m}}
$$

The speed grows as the square root of time, not in proportion to it: the force N/v is large at the start and falls as the car speeds up. The distance is the area under the speed-time curve:

$$s = \int_0^t \sqrt{\frac{2\,N\,t}{m}}\,dt = \frac{2}{3}\sqrt{\frac{2\,N}{m}}\,t^{3/2}$$

which is the same as two thirds of the final speed times the time.

$$\begin{aligned} v &= \sqrt{\frac{2NT}{m}} \\ &= \sqrt{\frac{2 \times 16 \times 4}{2}} \\ &= 8\ \mathrm{m/s} \\ s &= \frac{2vT}{3} \\ &= \frac{2 \times 8 \times 4}{3} \\ &= \frac{64}{3}\ \mathrm{m} \end{aligned}$$

> Result.
>
> After 4 s the car moves at 8 m/s and has covered $\frac{64}{3}$ m, about 21 m. A car with steady acceleration reaching the same speed in the same time would have covered only 16 m; here the early burst of speed carries it farther.

Question 12. Constant power: speeding up between two speeds

A car of mass 1000 kg moves on a level road with its engine delivering a constant 40000 W to the wheels. Resistance is negligible. Find (a) the time it takes to go from 10 m/s to 20 m/s, (b) the distance it covers meanwhile, and (c) its acceleration at each of the two speeds.

Solution

(a) In a time T the engine does the work $N\,T$, and this is the gain in kinetic energy:

$$N\,T = \tfrac{1}{2}\,m\left(v_2^2 - v_1^2\right)$$

$$\begin{aligned} T &= \frac{m(v_2^2 - v_1^2)}{2N} \\ &= \frac{1000 \times (20^2 - 10^2)}{2 \times 40000} \\ &= 3.75\ \mathrm{s} \end{aligned}$$

(b) The driving force at speed v is N/v, so $m\,v\,\dfrac{dv}{ds} = \dfrac{N}{v}$. Separating, $ds = \dfrac{m\,v^2}{N}\,dv$, and adding up from the first speed to the second:

$$s = \frac{m\left(v_2^3 - v_1^3\right)}{3\,N}$$

$$
\begin{aligned}
s &= \frac{m(v_2^3 - v_1^3)}{3N} \\
&= \frac{1000 \times (20^3 - 10^3)}{3 \times 40000} \\
&= \frac{175}{3}\ \mathrm{m}
\end{aligned}
$$

(c) The acceleration is the force divided by the mass, $N/(m\,v)$:

$$
\begin{aligned}
a_1 &= \frac{N}{m v_1} \\
&= \frac{40000}{1000 \times 10} \\
&= 4\ \mathrm{m/s^2} \\
a_2 &= \frac{N}{m v_2} \\
&= \frac{40000}{1000 \times 20} \\
&= 2\ \mathrm{m/s^2}
\end{aligned}
$$

> Result.
>
> The car takes 3.75 s and covers $\frac{175}{3}$ m, about 58 m. Its acceleration falls from $4\ \mathrm{m/s^2}$ to $2\ \mathrm{m/s^2}$: at constant power, doubling the speed halves the pull.

Question 13. Constant power against resistance: the top speed

The car of Question 12 (mass 1000 kg, engine power 40000 W) now meets a steady resistance of 1600 N. Find its top speed on a level road, and its acceleration at the moment its speed is 10 m/s.

Solution

The driving force is N/v and falls as the speed rises. The car stops gaining speed when the driving force has dropped to the resistance:

$$
\begin{aligned}
\frac{N}{v_{\max}} &= R \\
v_m &= \frac{N}{R} \\
&= \frac{40000}{1600} \\
&= 25\ \mathrm{m/s}
\end{aligned}
$$

Below the top speed the driving force is larger than the resistance, and the difference accelerates the car:

$$
\begin{aligned}
F &= \frac{N}{v_1} \\
&= \frac{40000}{10} \\
&= 4000\,\text{N} \\
a &= \frac{F - R}{m} \\
&= \frac{4000 - 1600}{1000} \\
&= 2.4\,\text{m/s}^2
\end{aligned}
$$

> Result.
>
> The top speed is 25 m/s. At 10 m/s the driving force is 4000 N and the acceleration is $2.4\ \text{m/s}^2$.

Question 14. Filling a tank on a roof

A pump must fill a tank of volume $3\ \text{m}^3$ on a roof 12 m above the water level of a well, in 10 minutes. The water arrives in the tank with negligible speed. Find the useful power of the pump.

Solution

The pump's useful work is the gain in potential energy of the water lifted. The power is that work divided by the time, with the time in seconds:

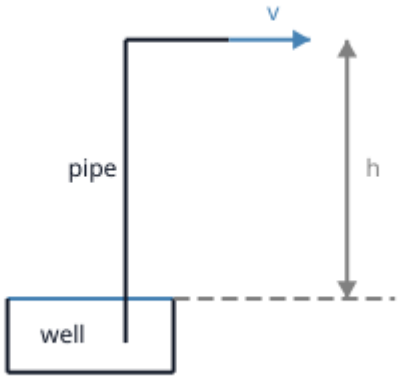
$$
P = \frac{M\,g\,h}{t}
$$

$$
\begin{aligned}
M &= \rho V \\
&= 1000 \times 3 \\
&= 3000\,\text{kg} \\
W &= Mgh \\
&= 3000 \times 10 \times 12 \\
&= 360000\,\text{J} \\
t_s &= 60T \\
&= 60 \times 10 \\
&= 600\,\text{s} \\
P &= \frac{W}{t_s} \\
&= \frac{360000}{600} \\
&= 600\,\text{W}
\end{aligned}
$$

> Result.
>
> The pump must deliver 600 W of useful power.

Question 15. A pump that also gives the water speed

A pump draws 600 litres of water a minute from a well and delivers it at a point 15 m above the water level, where the water leaves the pipe at 8 m/s. The pump turns 70% of the power supplied to it into useful power. Find the useful power and the power that must be supplied.



Solution

Each kilogram of water gains potential energy $g\,h$ and kinetic energy $\frac{1}{2}v^2$. With μ kilograms passing each second, the useful power is

$$P = \mu\left(g\,h + \tfrac{1}{2}\,v^2\right)$$

One litre of water has a mass of 1 kg, so the rate in kilograms per second is the rate in litres per minute divided by 60:

$$
\begin{aligned}
r &= \frac{V}{60} \\
&= \frac{600}{60} \\
&= 10\,\mathrm{kg/s} \\
U &= gh \\
&= 10 \times 15 \\
&= 150\,\mathrm{J/kg} \\
K &= \frac{v^2}{2} \\
&= \frac{8^2}{2} \\
&= 32\,\mathrm{J/kg} \\
P &= r(U + K) \\
&= 10 \times (150 + 32) \\
&= 1820\,\mathrm{W} \\
S &= \frac{100P}{e} \\
&= \frac{100 \times 1820}{70} \\
&= 2600\,\mathrm{W}
\end{aligned}
$$

> Result.
>
> The useful power is 1820 W, of which lifting takes 1500 W and the speed of the water 320 W. The pump must be supplied with 2600 W.

Question 16. Water from a hose

Water leaves a fire hose through a nozzle of cross-section $5\ \mathrm{cm}^2$ at 20 m/s. Find the power carried by the jet, that is, the kinetic energy it carries away each second. By what factor does this power change if the speed is doubled?

Solution

In one second a column of water of length v leaves the nozzle. With the cross-section written S, its volume is $S\,v$ and its mass is $\rho\,S\,v$. Each kilogram carries the kinetic energy $\tfrac{1}{2}v^2$, so

$$P = (\rho\,S\,v)\cdot\tfrac{1}{2}\,v^2 = \tfrac{1}{2}\,\rho\,S\,v^3$$

The cross-section must be in square metres: 1 cm^2 is $10^{-4}\ \mathrm{m}^2$.

$$
\begin{aligned}
S &= \frac{a}{10000} \\
&= \frac{5}{10000} \\
&= \frac{1}{2000}\,\mathrm{m}^2 \\
r &= \rho S v \\
&= 1000 \times \frac{1}{2000} \times 20 \\
&= 10\,\mathrm{kg/s} \\
P &= \frac{r v^2}{2} \\
&= \frac{10 \times 20^2}{2} \\
&= 2000\,\mathrm{W}
\end{aligned}
$$

At twice the speed, twice as much water leaves each second and each kilogram carries four times the energy:

$$
\begin{aligned}
w &= 2v \\
&= 2 \times 20 \\
&= 40\,\mathrm{m/s} \\
P_2 &= \frac{\rho S w^3}{2} \\
&= \frac{1000 \times \frac{1}{2000} \times 40^3}{2} \\
&= 16000\,\mathrm{W} \\
f &= \frac{P_2}{P} \\
&= \frac{16000}{2000} \\
&= 8
\end{aligned}
$$

> Result.
>
> The jet carries 2000 W. Doubling the speed multiplies the power by 8: the power of a jet goes as the cube of its speed.

Question 17. Water falling on a turbine

Water flows down a pipe from a reservoir to a turbine 30 m below the water surface, at the rate of 2 m^3 a second. The turbine and generator together turn 80% of the power of the falling water into electrical power. Find the electrical power.

Solution

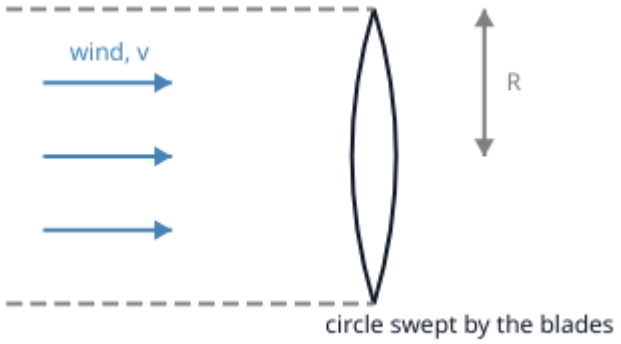
Each second a mass $\rho\,Q$ of water drops through the height h and gives up the potential energy $\rho\,Q\,g\,h$. That is the power available; the set delivers the stated share of it:

$$\begin{aligned} r &= \rho Q \\ &= 1000 \times 2 \\ &= 2000\,\mathrm{kg/s} \\ P &= rgh \\ &= 2000 \times 10 \times 30 \\ &= 600000\,\mathrm{W} \\ E &= \frac{Pe}{100} \\ &= \frac{600000 \times 80}{100} \\ &= 480000\,\mathrm{W} \end{aligned}$$

> Result.
>
> The falling water supplies 600 kW, and the set delivers 480 kW of electrical power.

Question 18. A windmill

The blades of a windmill sweep a circle of radius 4 m, face on to a wind of 10 m/s. The density of air is $1.2\ \mathrm{kg/m^3}$. The windmill turns 25% of the kinetic energy of the air passing through the circle into useful work. Find its useful power. What does it become if the wind speed falls to half?



Solution

Treat the wind as a jet whose cross-section is the circle swept by the blades, $S = \pi R^2$. Each second a column of air of length v passes through it, of mass $\rho\,S\,v$, carrying the kinetic energy

$$P_{\text{wind}} = \tfrac{1}{2}\,\rho\,S\,v^3$$

$$
\begin{aligned}
S &= \pi R^2 \\
&= \pi 4^2 \\
&= 50.26548246\,\mathrm{m}^2 \\
r &= \rho S v \\
&= 1.2 \times 50.26548246 \times 10 \\
&= 603.1857895\,\mathrm{kg/s} \\
P &= \frac{r v^2}{2} \\
&= \frac{603.1857895 \times 10^2}{2} \\
&= 30159.28947\,\mathrm{W} \\
E &= \frac{P e}{100} \\
&= \frac{30159.28947 \times 25}{100} \\
&= 7539.822369\,\mathrm{W}
\end{aligned}
$$

At half the wind speed, half as much air arrives each second and each kilogram carries a quarter of the energy:

$$
\begin{aligned}
w &= \frac{v}{2} \\
&= \frac{10}{2} \\
&= 5\,\mathrm{m/s} \\
P_2 &= \frac{\rho S w^3}{2} \\
&= \frac{1.2 \times 50.26548246 \times 5^3}{2} \\
&= 3769.911184\,\mathrm{W} \\
E_2 &= \frac{P_2 e}{100} \\
&= \frac{3769.911184 \times 25}{100} \\
&= 942.4777961\,\mathrm{W} \\
f &= \frac{E}{E_2} \\
&= \frac{7539.822369}{942.4777961} \\
&= 8
\end{aligned}
$$

> Result.
>
> The wind brings about 30 kW through the circle, and the windmill delivers about 7540 W. At half the wind speed it delivers about 942 W, one part in 8: a windmill's power goes as the cube of the wind speed.

Question 19. Power of friction on a sliding block

A block of mass 2 kg is sent sliding across a rough level floor at 10 m/s. The coefficient of friction is 0.25. Find (a) the power of the friction force 1 s after the start, and (b) its average power over the whole slide, until the block stops.

Solution

On a level floor the normal force equals the weight, so the friction force is $k\,m\,g$. It points against the velocity, so its power is negative:

$$P = f \cdot v = -\,k\,m\,g\,v$$

(a) Friction is the only horizontal force. It slows the block at the steady rate $k\,g$:

$$\begin{aligned} f &= kmg \\ &= 0.25 \times 2 \times 10 \\ &= 5\,\mathrm{N} \\ a &= kg \\ &= 0.25 \times 10 \\ &= 2.5\,\mathrm{m/s^2} \\ v &= u - aT \\ &= 10 - 2.5 \times 1 \\ &= 7.5\,\mathrm{m/s} \\ P &= -fv \\ &= -5 \times 7.5 \\ &= -37.5\,\mathrm{W} \end{aligned}$$

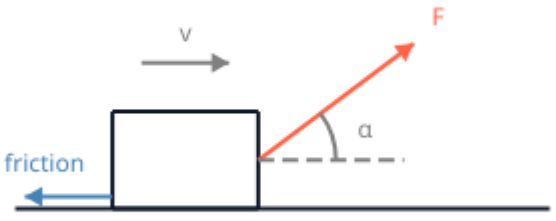
(b) The block stops when its speed reaches zero. Over the slide friction does work minus the friction force times the distance:

$$
\begin{aligned}
T_s &= \frac{u}{a} \\
&= \frac{10}{2.5} \\
&= 4\,\mathrm{s} \\
s &= \frac{uT_s}{2} \\
&= \frac{10 \times 4}{2} \\
&= 20\,\mathrm{m} \\
W &= -fs \\
&= -5 \times 20 \\
&= -100\,\mathrm{J} \\
A &= \frac{W}{T_s} \\
&= \frac{-100}{4} \\
&= -25\,\mathrm{W}
\end{aligned}
$$

> Result.
>
> After 1 s friction has a power of -37.5 W. Over the whole slide of 4 s its average power is -25 W. The size of the power is greatest at the start, when the block is fastest, and falls steadily to zero; all of this energy becomes heat.

Question 20. Dragging a crate with a rope at an angle

A crate of mass 44 kg is dragged across a rough level floor at a steady 2 m/s by a rope that slopes upward. The rope makes an angle α with the floor, with $\sin\alpha$ equal to $3/5$ and $\cos\alpha$ equal to $4/5$. The coefficient of friction is 0.5. Find the pull of the rope, the power it delivers, and the power of the friction force.



Solution

The speed is steady, so the forces balance in both directions. The upward part of the pull takes some weight off the floor, which lowers the normal force and with it the friction:

$$N = m\,g - F\sin\alpha$$

$$F\cos\alpha = k\,N = k\,(m\,g - F\sin\alpha)$$

Solving for the pull:

$$F = \frac{k\,m\,g}{\cos\alpha + k\sin\alpha}$$

$$\begin{aligned} F &= \frac{kmg}{\frac{q}{r} + \frac{kp}{r}} \\ &= \frac{0.5 \times 44 \times 10}{\frac{4}{5} + \frac{0.5\times 3}{5}} \\ &= 200\,\mathrm{N} \end{aligned}$$

$$\begin{aligned} N &= mg - \frac{Fp}{r} \\ &= 44 \times 10 - \frac{200 \times 3}{5} \\ &= 320\,\mathrm{N} \end{aligned}$$

$$\begin{aligned} f &= kN \\ &= 0.5 \times 320 \\ &= 160\,\mathrm{N} \end{aligned}$$

Only the part of the pull along the motion delivers power. Friction points against the motion:

$$\begin{aligned} P &= F\frac{q}{r}v \\ &= 200 \times \frac{4}{5} \times 2 \\ &= 320\,\mathrm{W} \end{aligned}$$

$$\begin{aligned} P_f &= -fv \\ &= -160 \times 2 \\ &= -320\,\mathrm{W} \end{aligned}$$

> Result.
>
> The rope pulls with 200 N and delivers 320 W. The friction force is 160 N and its power is -320 W. The two powers add up to zero: at steady speed everything the rope supplies is turned into heat by friction.

Question 21. Air resistance and the power for a higher speed

At high speed the main force against a car is air resistance, which grows as the square of the speed: $F = c\,v^2$. A car needs 54000 W at the wheels to hold a steady 30 m/s on a level road. Taking air resistance as the only resisting force, find the constant c, and the power needed to hold 40 m/s.

Solution

At steady speed the driving force equals the air resistance, so the power is

$$P = F\,v = c\,v^3$$

The power needed grows as the cube of the speed. From the known speed and power:

$$\begin{aligned} c &= \frac{N}{v_1^3} \\ &= \frac{54000}{30^3} \\ &= 2\ \mathrm{kg/m} \\ F_1 &= c v_1^2 \\ &= 2 \times 30^2 \\ &= 1800\ \mathrm{N} \\ P_2 &= c v_2^3 \\ &= 2 \times 40^3 \\ &= 128000\ \mathrm{W} \\ F_2 &= c v_2^2 \\ &= 2 \times 40^2 \\ &= 3200\ \mathrm{N} \\ f &= \frac{P_2}{N} \\ &= \frac{128000}{54000} \\ &= \frac{64}{27} \end{aligned}$$

> Result.
>
> The constant is 2 kg/m. Holding 40 m/s needs 128 kW against 54 kW at 30 m/s: $\frac{64}{27}$ times the power, well over double, for a speed only one third higher. The resisting force rises from 1800 N to 3200 N.

Question 22. A parachutist: gravity against air resistance

A parachutist of total mass 80 kg comes down at a steady 5 m/s. The air resistance on the open parachute is proportional to the square of the speed, $F = c\,v^2$. Find (a) the power of gravity and of the air resistance during the steady descent, and (b) the same two powers at a moment when the speed is half of the steady value, and what the difference between them does.

Solution

(a) At steady speed the air resistance equals the weight. Gravity acts along the motion and the air resistance against it:

$$\begin{aligned} W &= mg \\ &= 80 \times 10 \\ &= 800\,\mathrm{N} \\ P_g &= W v_t \\ &= 800 \times 5 \\ &= 4000\,\mathrm{W} \\ P_a &= -W v_t \\ &= -800 \times 5 \\ &= -4000\,\mathrm{W} \end{aligned}$$

The two cancel: the potential energy lost each second all goes into the air.

(b) The constant follows from the steady descent, $c\,v_t^2 = m\,g$. At half the speed the air resistance is only a quarter of the weight:

$$\begin{aligned} c &= \frac{W}{v_t^2} \\ &= \frac{800}{5^2} \\ &= 32\,\mathrm{kg/m} \\ w &= \frac{v_t}{2} \\ &= \frac{5}{2} \\ &= \frac{5}{2}\,\mathrm{m/s} \\ F &= cw^2 \\ &= 32 \times (\frac{5}{2})^2 \\ &= 200\,\mathrm{N} \\ Q_g &= Ww \\ &= 800 \times \frac{5}{2} \\ &= 2000\,\mathrm{W} \\ Q_a &= -Fw \\ &= -200 \times \frac{5}{2} \\ &= -500\,\mathrm{W} \\ Q &= Q_g + Q_a \\ &= 2000 + (-500) \\ &= 1500\,\mathrm{W} \end{aligned}$$

> Result.
>
> In steady descent gravity delivers 4000 W and the air resistance -4000 W. At 2.5 m/s gravity delivers 2000 W and the air resistance -500 W; the remaining 1500 W goes into kinetic energy, so the parachutist is still speeding up.

Question 23. A cyclist against rolling and air resistance

A cyclist rides on a level road at a steady 10 m/s. The rolling resistance is a steady 10 N, and the air resistance is $c\,v^2$ with c equal to 0.25 kg/m. (a) Find the power the cyclist must deliver, and how it is shared between the two resistances. (b) The cyclist and cycle have a mass of 80 kg. Find the power needed to keep the same speed up a road that rises 1 m for every 50 m travelled.

Solution

(a) At steady speed the driving force equals the total resistance, and the power is that force times the speed:

$$
\begin{aligned}
P &= \left(R + c\,v^2\right) v \\
F_a &= cv^2 \\
&= 0.25 \times 10^2 \\
&= 25\,\mathrm{N} \\
P_r &= Rv \\
&= 10 \times 10 \\
&= 100\,\mathrm{W} \\
P_a &= F_a v \\
&= 25 \times 10 \\
&= 250\,\mathrm{W} \\
P &= P_r + P_a \\
&= 100 + 250 \\
&= 350\,\mathrm{W}
\end{aligned}
$$

(b) On the slope the part of the weight along the road is added to the resistances:

$$
\begin{aligned}
G &= \frac{mg}{n} \\
&= \frac{80 \times 10}{50} \\
&= 16\,\mathrm{N} \\
P_s &= (R + F_a + G)v \\
&= (10 + 25 + 16) \times 10 \\
&= 510\,\mathrm{W}
\end{aligned}
$$

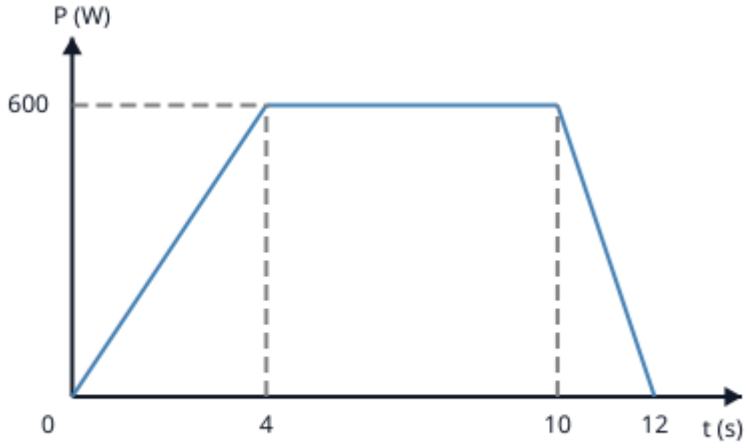
> Result.
>
> On the level the cyclist delivers 350 W: 100 W against rolling resistance and 250 W against the air. Up the slope the same speed needs 510 W. The air's share goes as the cube of the speed, so it dominates at high speed and matters little at low speed.

Power, part 2: questions 24 to 44

Graphs, greatest power, motion in a circle, a spring, more on friction and conveyor belts, more on air resistance and constant power, flowing mass, and machines and units. Some questions have numbers and some are worked in symbols. Every number in a solution is worked out by ApasScript. Where it is needed, take the acceleration of free fall as 10 $\mathrm{m/s^2}$ and the density of water as 1000 $\mathrm{kg/m^3}$.

Question 24. Work from a power-time graph

A motor drives a trolley of mass 12 kg from rest along a smooth level track. The power it delivers changes with time as in the graph: it rises steadily to 600 W in the first 4 s, stays there until 10 s, and falls steadily to zero at 12 s. Find the work done by the motor, its average power, and the final speed of the trolley.



Solution

Power is the rate of doing work, so the work done is the area under the power-time graph:

$$W = \int P \, dt$$

The area splits into a triangle, a rectangle and a second triangle:

$$
\begin{aligned}
W_1 &= \frac{P_0 T_1}{2} \\
&= \frac{600 \times 4}{2} \\
&= 1200\,\mathrm{J} \\
W_2 &= P_0(T_2 - T_1) \\
&= 600 \times (10 - 4) \\
&= 3600\,\mathrm{J} \\
W_3 &= \frac{P_0(T_3 - T_2)}{2} \\
&= \frac{600 \times (12 - 10)}{2} \\
&= 600\,\mathrm{J} \\
W &= W_1 + W_2 + W_3 \\
&= 1200 + 3600 + 600 \\
&= 5400\,\mathrm{J} \\
A &= \frac{W}{T_3} \\
&= \frac{5400}{12} \\
&= 450\,\mathrm{W}
\end{aligned}
$$

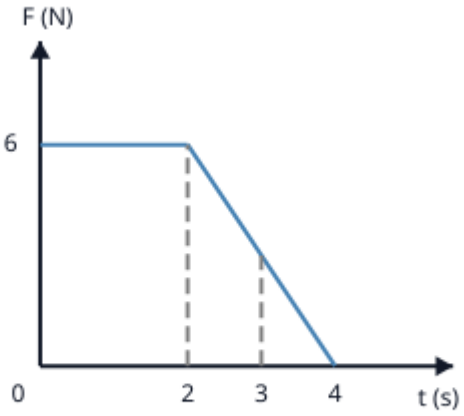
The track is smooth and level, so all this work becomes kinetic energy of the trolley:

$$
\begin{aligned}
v &= \sqrt{\frac{2W}{m}} \\
&= \sqrt{\frac{2 \times 5400}{12}} \\
&= 30\,\mathrm{m/s}
\end{aligned}
$$

> Result.
>
> The motor does 5400 J of work, an average of 450 W over the 12 s, and the trolley ends at 30 m/s.

Question 25. Power from a force-time graph

A block of mass 2 kg rests on a smooth level table. A horizontal force acts on it as in the graph: it is a steady 6 N for the first 2 s and then falls steadily to zero at 4 s. Find the power delivered by the force at 2 s and at 3 s. What is the power at 4 s?



Solution

Power is force times speed, and both are needed at the same moment. The force is read off the graph. The speed comes from the area under the force-time graph, which is the momentum given to the block:

$$m\,v = \int F\,dt$$

The first moment, 2 s. The area so far is a rectangle:

$$\begin{aligned} v_1 &= \frac{F_0 T_1}{m} \\ &= \frac{6 \times 2}{2} \\ &= 6\ \mathrm{m/s} \\ P_1 &= F_0 v_1 \\ &= 6 \times 6 \\ &= 36\ \mathrm{W} \end{aligned}$$

The second moment, 3 s. The force has fallen part of the way along the sloping line. The area added since 2 s is a trapezium:

$$
\begin{aligned}
F_3 &= \frac{F_0(T_2 - T_3)}{T_2 - T_1} \\
&= \frac{6 \times (4 - 3)}{4 - 2} \\
&= 3\,\mathrm{N} \\
J &= \frac{(F_0 + F_3)(T_3 - T_1)}{2} \\
&= \frac{(6 + 3) \times (3 - 2)}{2} \\
&= 4.5\,\mathrm{Ns} \\
v_3 &= v_1 + \frac{J}{m} \\
&= 6 + \frac{4.5}{2} \\
&= 8.25\,\mathrm{m/s} \\
P_3 &= F_3 v_3 \\
&= 3 \times 8.25 \\
&= 24.75\,\mathrm{W}
\end{aligned}
$$

The end, 4 s. The block is at its fastest, but the force is zero, so the power is zero:

$$
\begin{aligned}
v_2 &= v_1 + \frac{F_0(T_2 - T_1)}{2m} \\
&= 6 + \frac{6 \times (4 - 2)}{2 \times 2} \\
&= 9\,\mathrm{m/s} \\
P_2 &= 0 v_2 \\
&= 0 \times 9 \\
&= 0\,\mathrm{W}
\end{aligned}
$$

> Result.
>
> The power is 36 W at 2 s, 24.75 W at 3 s and 0 W at 4 s, although the block is then moving at 9 m/s. A large speed gives no power without a force.

Question 26. Power that grows in proportion to time

A body of mass m starts from rest on a smooth level surface. The power delivered to it grows in proportion to time, $P = ct$, where c is a constant. Find its speed and the distance covered as functions of time, and the force acting on it.

Solution

The work done up to the time t is the area under the power-time line, a triangle:

$$
W = \int_0^t c\,t\,dt = \tfrac{1}{2}\,c\,t^2
$$

It all becomes kinetic energy:

$$\tfrac{1}{2}\,m\,v^2 = \tfrac{1}{2}\,c\,t^2$$

$$v = t\,\sqrt{\frac{c}{m}}$$

The speed is proportional to time, so the acceleration is constant, $\sqrt{c/m}$. The distance is then that of uniformly accelerated motion, and the force is mass times this acceleration:

> Result.
>
> $$v = t\,\sqrt{\frac{c}{m}}$$
>
> $$s = \tfrac{1}{2}\,t^2\,\sqrt{\frac{c}{m}}$$
>
> $$F = \sqrt{c\,m}$$
>
> Power growing in proportion to time is exactly what a constant force gives to a body starting from rest (compare Question 1).

Question 27. Where a falling thrust delivers the greatest power

The sledge of Question 5 has a mass of 2 kg and starts from rest at $x = 0$ on smooth ice, pushed by the force $F = F_0 - k\,x$ with $F_0 = 20$ N and k equal to 2 N/m. At what position does the force deliver the greatest power, and how large is it?

Solution

Early on the force is large but the sledge is slow; later the sledge is fast but the force is small. The power $F\,v$ is greatest somewhere in between. From the work done so far,

$$\tfrac{1}{2}\,m\,v^2 = F_0\,x - \tfrac{1}{2}\,k\,x^2 = \frac{F_0^2 - F^2}{2\,k}$$

where the last step uses $F = F_0 - k\,x$. So the square of the power is

$$P^2 = F^2\,v^2 = \frac{F^2\left(F_0^2 - F^2\right)}{k\,m}$$

The two factors F^2 and $F_0^2 - F^2$ always add up to F_0^2. A product of two numbers with a fixed sum is greatest when they are equal, so the power is greatest when

$$F^2 = \tfrac{1}{2}\,F_0^2$$

that is, when the force has fallen to $F_0/\sqrt{2}$. The position and the power follow:

$$x_{\max} = \frac{F_0}{k}\left(1 - \frac{1}{\sqrt{2}}\right)$$

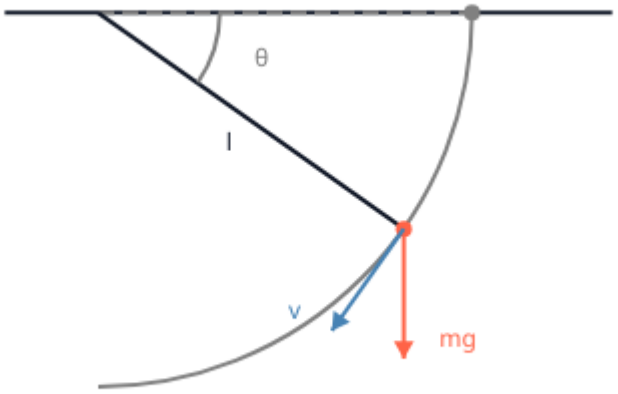
$$P_{\max} = \frac{F_0^2}{2\sqrt{k\,m}}$$

$$
\begin{aligned}
P_m &= \frac{F_0^2}{2\sqrt{km}} \\
&= \frac{20^2}{2\sqrt{2 \times 2}} \\
&= 100\ \text{W}
\end{aligned}
$$

> Result.
>
> The power is greatest at x = 2.928932 m, where it is 100 W. At 4 m it had already dropped to 96 W (Question 5).

Question 28. A pendulum bob: where gravity delivers the greatest power

A bob of mass m hangs on a light string of length l. It is held with the string horizontal and released from rest. At what position during the swing down is the power of gravity greatest, and what is that power?



Solution

Let the string have turned through the angle θ below the horizontal. The bob has dropped a height $l \sin\theta$, so its speed is

$$v = \sqrt{2\,g\,l\,\sin\theta}$$

The velocity is along the circle, at right angles to the string. The part of the weight along the velocity is $m\,g\cos\theta$. (The tension is along the string, at right angles to the velocity, and delivers no power.) So

$$P = m\,g\cos\theta\,\sqrt{2\,g\,l\,\sin\theta}$$

At the start the speed is zero, and at the bottom the weight is at right angles to the velocity: the power is zero at both ends and greatest in between. Squaring, the power is greatest when $\cos^2\theta\,\sin\theta$ is greatest. Write s for $\sin\theta$; then this is $s\,(1 - s^2)$, and its slope is zero when

$$1 - 3\,s^2 = 0$$

Result.

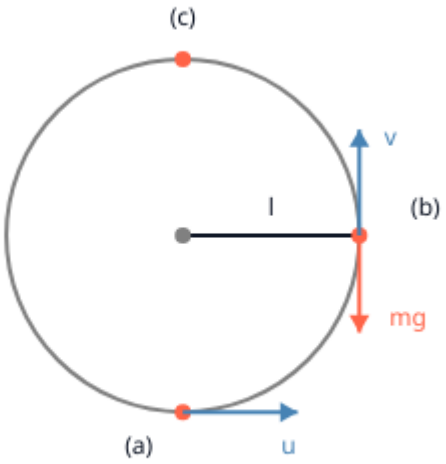
$$\sin\theta = \frac{1}{\sqrt{3}}$$

$$P_{\max} = m\,g\sqrt{\frac{2}{3}}\sqrt{\frac{2\,g\,l}{\sqrt{3}}} = \frac{2\,m\,g\sqrt{g\,l}}{3^{3/4}}$$

The string is then about $35°$ below the horizontal, that is about $55°$ from the vertical.

Question 29. A bob in a vertical circle

A bob of mass 0.5 kg whirls in a vertical circle on a light string 1.6 m long. It passes the lowest point at 9 m/s. Find the power of the tension, and the power of gravity (a) at the lowest point, (b) when the string is horizontal and the bob is rising, and (c) at the highest point.



Solution

The tension always points along the string, and the velocity is always along the circle, at right angles to the string. So the power of the tension is zero at every point: it turns the bob but never changes its energy.

(a) and (c) At the lowest and highest points the velocity is horizontal and gravity is vertical. They are at right angles, so the power of gravity is zero there too.

(b) With the string horizontal the bob has risen by l and is moving straight up, directly against gravity. Its speed comes from energy conservation:

$$
\begin{aligned}
v &= \sqrt{u^2 - 2gl} \\
&= \sqrt{9^2 - 2 \times 10 \times 1.6} \\
&= 7\,\mathrm{m/s} \\
P &= -mgv \\
&= -0.5 \times 10 \times 7 \\
&= -35\,\mathrm{W}
\end{aligned}
$$

The string stays taut over the top only if the speed there is at least $\sqrt{g\,l}$:

$$
\begin{aligned}
V &= u^2 - 4gl \\
&= 9^2 - 4 \times 10 \times 1.6 \\
&= 17\,\mathrm{m^2/s^2} \\
G &= gl \\
&= 10 \times 1.6 \\
&= 16\,\mathrm{m^2/s^2}
\end{aligned}
$$

> Result.
>
> The power of the tension is zero everywhere. The power of gravity is zero at the lowest and highest points and -35 W when the string is horizontal on the way up (and the same with a plus sign on the way down). The square of the speed at the top, 17, is more than 16, so the string stays taut.

Question 30. A block on a spring: the greatest power of the spring force

A block of mass 1 kg on a smooth level table is fixed to a spring of stiffness 100 N/m. The spring is stretched by 0.2 m and the block is released from rest. Find the greatest power the spring force delivers to the block on its way back to the unstretched position, and the stretch at which this happens.

Solution

Write A for the first stretch. When the stretch has fallen to x, the spring force is $k\,x$, pulling along the motion, and the speed follows from the energy the spring has given up:

$$\tfrac{1}{2}\,m\,v^2 = \tfrac{1}{2}\,k\left(A^2 - x^2\right)$$

The power is $P = k\,x\,v$, so

$$P^2 = \frac{k^3}{m}\,x^2\left(A^2 - x^2\right)$$

The power is zero at the start (no speed) and at the unstretched position (no force). The two factors x^2 and $A^2 - x^2$ add up to A^2, so their product is greatest when they are equal: $x = A/\sqrt{2}$. There

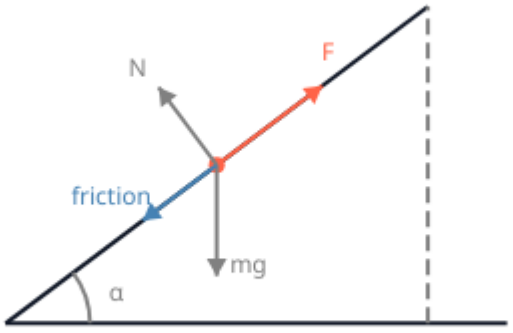
$$P_{\max} = \tfrac{1}{2}\,k\,A^2\sqrt{\frac{k}{m}}$$

$$
\begin{aligned}
P_m &= \frac{kA^2}{2}\sqrt{\frac{k}{m}} \\
&= \frac{100 \times 0.2^2}{2} \times \sqrt{\frac{100}{1}} \\
&= 20\,\mathrm{W}
\end{aligned}
$$

> Result.
>
> The spring force delivers at most 20 W, when the stretch has fallen to 0.141421 m. The same holds for any oscillating block: the power is greatest half way in energy, not half way in distance.

Question 31. Pulling a block up a rough incline

A block of mass 10 kg is pulled up a rough incline at a steady 2 m/s by a rope parallel to the incline. The incline has $\sin\alpha$ equal to $3/5$ and $\cos\alpha$ equal to $4/5$, and the coefficient of friction is 0.25. Find the power of each force on the block: the pull, gravity, friction and the normal force.



Solution

The speed is steady, so the forces balance. Across the incline the normal force balances the part of the weight pressing on it; along the incline the pull balances the part of the weight down the slope and the friction:

$$
\begin{aligned}
G &= \frac{mgp}{r} \\
&= \frac{10 \times 10 \times 3}{5} \\
&= 60\,\mathrm{N} \\
N &= \frac{mgq}{r} \\
&= \frac{10 \times 10 \times 4}{5} \\
&= 80\,\mathrm{N} \\
f &= kN \\
&= 0.25 \times 80 \\
&= 20\,\mathrm{N} \\
F &= G + f \\
&= 60 + 20 \\
&= 80\,\mathrm{N}
\end{aligned}
$$

The pull is along the velocity; gravity's slope part and friction are against it; the normal force is at right angles to it:

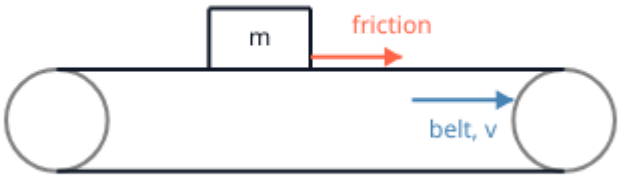
$$
\begin{aligned}
P_F &= Fv \\
&= 80 \times 2 \\
&= 160\,\mathrm{W} \\
P_g &= -Gv \\
&= -60 \times 2 \\
&= -120\,\mathrm{W} \\
P_f &= -fv \\
&= -20 \times 2 \\
&= -40\,\mathrm{W} \\
P_N &= 0N \\
&= 0 \times 80 \\
&= 0\,\mathrm{W}
\end{aligned}
$$

> Result.
>
> The pull delivers 160 W. The power of gravity is -120 W: that energy is stored as potential energy. The power of friction is -40 W: that energy becomes heat. The normal force delivers 0 W. The four add up to zero, as they must at steady speed.

Question 32. A block dropped on a moving conveyor belt

A conveyor belt runs at a steady 4 m/s. A block of mass 5 kg is lowered gently onto it with no speed of its own. The coefficient of friction between block and belt is 0.2. Find (a) how long the block slips, (b) the extra power the motor must give while it slips, (c) the extra work done by the motor, and where

that work goes.



Solution

While the block is slower than the belt, the belt drags it forward with the friction force $k\,m\,g$. That force speeds the block up at the rate $k\,g$ until it moves with the belt:

$$\begin{aligned} f &= kmg \\ &= 0.2 \times 5 \times 10 \\ &= 10\,\mathrm{N} \\ a &= kg \\ &= 0.2 \times 10 \\ &= 2\,\mathrm{m/s^2} \\ T &= \frac{v}{a} \\ &= \frac{4}{2} \\ &= 2\,\mathrm{s} \end{aligned}$$

(b) The block pulls back on the belt with the same force. To keep the belt at its speed the motor must supply that force at the belt's speed:

$$\begin{aligned} P &= fv \\ &= 10 \times 4 \\ &= 40\,\mathrm{W} \end{aligned}$$

(c) In the slipping time the belt moves on, while the block, starting from rest, covers only half as far:

$$
\begin{aligned}
s_c &= vT \\
&= 4 \times 2 \\
&= 8\,\mathrm{m} \\
s_b &= \frac{vT}{2} \\
&= \frac{4 \times 2}{2} \\
&= 4\,\mathrm{m} \\
W &= f s_c \\
&= 10 \times 8 \\
&= 80\,\mathrm{J} \\
K &= f s_b \\
&= 10 \times 4 \\
&= 40\,\mathrm{J} \\
Q &= f(s_c - s_b) \\
&= 10 \times (8 - 4) \\
&= 40\,\mathrm{J}
\end{aligned}
$$

> Result.
>
> The block slips for 2 s, and meanwhile the motor gives an extra 40 W. Its extra work is 80 J: 40 J becomes kinetic energy of the block and 40 J becomes heat at the rubbing surfaces. In symbols the motor's extra work is $m\,v^2$, twice the kinetic energy gained, whatever the coefficient of friction: a rougher belt only makes it happen sooner.

Question 33. Sand falling on a conveyor belt

Sand drops straight down from a hopper onto a level conveyor belt at the rate of 20 kg a second. The belt runs at a steady 3 m/s. Find the force needed to keep the belt moving at that speed and the power the motor must supply for it. How much of this power ends up as kinetic energy of the sand?

Solution

Write μ for the mass arriving each second. Every second that mass must be given the horizontal momentum $\mu\,v$. Force is the rate of change of momentum, so the belt must push the sand forward, and the motor must drive the belt, with the force

$$F = \mu\,v$$

The belt moves at v, so the power is

$$P = F\,v = \mu\,v^2$$

$$
\begin{aligned}
F &= rv \\
&= 20 \times 3 \\
&= 60\,\mathrm{N} \\
P &= Fv \\
&= 60 \times 3 \\
&= 180\,\mathrm{W} \\
K &= \frac{rv^2}{2} \\
&= \frac{20 \times 3^2}{2} \\
&= 90\,\mathrm{W} \\
Q &= P - K \\
&= 180 - 90 \\
&= 90\,\mathrm{W}
\end{aligned}
$$

> Result.
>
> The belt needs a force of 60 N and the motor supplies 180 W. Only 90 W becomes kinetic energy of the sand. The other 90 W, exactly half, becomes heat as each grain slips on the belt before it reaches the belt's speed, just as for the block of Question 32.

Question 34. A boat coasting against a resistance proportional to speed

A boat of mass m is moving at speed u when its engine is cut. The water resists it with a force proportional to its speed, $F = b\,v$. Find the power of the resistance when the boat has coasted a distance x, the distance in which it comes to rest, and the total work done by the resistance.

Solution

The only horizontal force is the resistance, so $m\,\dfrac{dv}{dt} = -b\,v$. Writing the acceleration as $v\,\dfrac{dv}{dx}$ and cancelling v:

$$
m\,\frac{dv}{dx} = -b
$$

The speed falls by the same amount in every metre:

$$
v = u - \frac{b\,x}{m}
$$

The resistance points against the velocity, so its power is $-b\,v^2$. The boat stops where the speed reaches zero, and the total work is the area under the force-distance line, a triangle of height $b\,u$:

Result.

$$P = -\,b\left(u - \frac{b\,x}{m}\right)^{2}$$

$$x_{\text{stop}} = \frac{m\,u}{b}$$

$$W = -\tfrac{1}{2}\,b\,u \cdot \frac{m\,u}{b} = -\tfrac{1}{2}\,m\,u^{2}$$

The resistance takes away exactly the kinetic energy the boat had. In time the speed dies away as $u\,e^{-bt/m}$ and the power as $-b\,u^{2}\,e^{-2bt/m}$: the boat creeps towards the stopping distance without ever quite reaching it.

Question 35. Constant power against air resistance: top speed

A car of mass 800 kg has an engine that delivers a constant 16000 W to the wheels. The only resistance is that of the air, $c\,v^{2}$ with c equal to 0.25 kg/m. Find the top speed of the car on a level road and its acceleration at 20 m/s.

Solution

Write N for the engine power. The driving force is N/v. At the top speed it equals the air resistance:

$$\frac{N}{v} = c\,v^{2}$$

$$v_{\max} = \sqrt[3]{\frac{N}{c}}$$

$$\begin{aligned} v_{m} &= \left(\frac{N}{c}\right)^{\frac{1}{3}} \\ &= \left(\frac{16000}{0.25}\right)^{\frac{1}{3}} \\ &= 40\,\text{m/s} \end{aligned}$$

Below the top speed the difference between the driving force and the resistance accelerates the car:

$$
\begin{aligned}
F &= \frac{N}{v_1} \\
&= \frac{16000}{20} \\
&= 800\,\mathrm{N} \\
R &= c v_1^2 \\
&= 0.25 \times 20^2 \\
&= 100\,\mathrm{N} \\
a &= \frac{F - R}{m} \\
&= \frac{800 - 100}{800} \\
&= 0.875\,\mathrm{m/s^2}
\end{aligned}
$$

> Result.
>
> The top speed is 40 m/s. At 20 m/s the driving force is 800 N against 100 N of air resistance, and the acceleration is $0.875\ \mathrm{m/s^2}$. Because the top speed goes as the cube root of the power, eight times the power would only double it.

Question 36. The steepest road a car can climb at a given speed

A car of mass 1000 kg has an engine that can deliver 30000 W to the wheels. The resistance to its motion is a steady 500 N. Find the steepest road it can climb at a steady 15 m/s, as the rise in metres for every 100 m of road.

Solution

At full power and at this speed the driving force is N/v. On the steepest road all of it is used: it balances the resistance and the part of the weight along the slope, $m\,g\sin\alpha$:

$$
\frac{N}{v} = R + m\,g\sin\alpha
$$

$$
\begin{aligned}
F &= \frac{N}{v} \\
&= \frac{30000}{15} \\
&= 2000\,\mathrm{N} \\
G &= F - R \\
&= 2000 - 500 \\
&= 1500\,\mathrm{N} \\
s &= \frac{G}{mg} \\
&= \frac{1500}{1000 \times 10} \\
&= 0.15 \\
h &= 100s \\
&= 100 \times 0.15 \\
&= 15\,\mathrm{m}
\end{aligned}
$$

> Result.
>
> The driving force is 2000 N, of which 1500 N is free to hold the car against the slope. So $\sin\alpha$ is 0.15: the road may rise 15 m in every 100 m. A slower climb allows a steeper road, because the same power gives a larger force at a lower speed.

Question 37. Hauling up a hanging chain

A chain of length L and mass λ per metre hangs over the edge of a roof, and is hauled up onto the roof at a steady speed v. Find the power needed as a function of time, the total work, and the average power. Then put in L = 10 m, λ = 2 kg/m and v = 0.5 m/s, and find the power at the start and after 10 s.

Solution

After a time t a length $v\,t$ is on the roof and $L - v\,t$ still hangs. At steady speed the pull equals the weight of the hanging part, and the power is that pull times the speed:

$$P = \lambda\, g\, (L - v\,t)\, v$$

The power falls steadily from $\lambda\, g\, L\, v$ to zero in the time L/v, so the average is half the starting value and the work is the average times the time:

$$W = \tfrac{1}{2}\,\lambda\, g\, L\, v \cdot \frac{L}{v} = \tfrac{1}{2}\,\lambda\, g\, L^2$$

That is the weight of the whole chain lifted through $L/2$, the rise of its middle point. With the numbers:

$$
\begin{aligned}
P_0 &= qgLv \\
&= 2 \times 10 \times 10 \times 0.5 \\
&= 100\,\mathrm{W} \\
l_1 &= L - vT \\
&= 10 - 0.5 \times 10 \\
&= 5\,\mathrm{m} \\
P_1 &= qgl_1 v \\
&= 2 \times 10 \times 5 \times 0.5 \\
&= 50\,\mathrm{W} \\
T_e &= \frac{L}{v} \\
&= \frac{10}{0.5} \\
&= 20\,\mathrm{s} \\
W &= \frac{qgL^2}{2} \\
&= \frac{2 \times 10 \times 10^2}{2} \\
&= 1000\,\mathrm{J} \\
A &= \frac{W}{T_e} \\
&= \frac{1000}{20} \\
&= 50\,\mathrm{W}
\end{aligned}
$$

> Result.
>
> The power is 100 W at the start and 50 W after 10 s, when 5 m still hangs. The haul takes 20 s and 1000 J of work, an average of 50 W.

Question 38. Pumping a tank empty over its rim

An upright tank with a level base of area $2\ \mathrm{m}^2$ is full of water to a depth of 3 m. A pump empties it over the rim in 5 minutes. The water leaves with negligible speed. Find the work done by the pump and its average useful power.

Solution

Not all the water is lifted through the same height. A thin layer at depth x below the rim, of thickness dx, has the mass $\rho\,S\,dx$ and must be raised by x. Adding up over all the layers:

$$W = \int_0^{H} \rho\, g\, S\, x\, dx = \tfrac{1}{2}\,\rho\, g\, S\, H^2$$

This is the weight of all the water times $H/2$, the rise of its centre of mass.

$$
\begin{aligned}
M &= \rho S H \\
&= 1000 \times 2 \times 3 \\
&= 6000\,\mathrm{kg} \\
W &= \frac{\rho g S H^2}{2} \\
&= \frac{1000 \times 10 \times 2 \times 3^2}{2} \\
&= 90000\,\mathrm{J} \\
t_s &= 60T \\
&= 60 \times 5 \\
&= 300\,\mathrm{s} \\
P &= \frac{W}{t_s} \\
&= \frac{90000}{300} \\
&= 300\,\mathrm{W}
\end{aligned}
$$

> Result.
>
> The pump does 90000 J of work, an average useful power of 300 W. Taking the full depth of 3 m for all 6000 kg of water would give twice the right answer.

Question 39. A fire jet that must reach a given height

A jet of water pointed straight up from a nozzle of cross-section $4\ \mathrm{cm}^2$ must reach a height of 20 m. Air resistance is negligible. Find the speed at the nozzle and the power carried by the jet.

Solution

Each piece of water rises like a stone thrown upward, so the speed at the nozzle must be $\sqrt{2\,g\,h}$. With the cross-section written S, the mass leaving each second is $\rho\,S\,v$, and each kilogram carries $\tfrac{1}{2}v^2$:

$$P = \tfrac{1}{2}\,\rho\,S\,v^3 = \rho\,S\,g\,h\,\sqrt{2\,g\,h}$$

$$
\begin{aligned}
v &= \sqrt{2gh} \\
&= \sqrt{2 \times 10 \times 20} \\
&= 20\,\mathrm{m/s} \\
S &= \frac{a}{10000} \\
&= \frac{4}{10000} \\
&= \frac{1}{2500}\,\mathrm{m^2} \\
r &= \rho S v \\
&= 1000 \times \frac{1}{2500} \times 20 \\
&= 8\,\mathrm{kg/s} \\
P &= \frac{r v^2}{2} \\
&= \frac{8 \times 20^2}{2} \\
&= 1600\,\mathrm{W}
\end{aligned}
$$

> Result.
>
> The water must leave at 20 m/s, and the jet carries 1600 W. The power goes as $h^{3/2}$: to reach twice the height needs nearly three times the power.

Question 40. A machine gun

A machine gun fires 300 bullets a minute. Each bullet has a mass of 20 g and leaves the barrel at 400 m/s. Find the power the gun gives to the bullets, and the average force needed to hold the gun still.

Solution

The power is the kinetic energy given to one bullet times the number of bullets fired each second (the first line below):

$$
\begin{aligned}
r &= \frac{n}{60} \\
&= \frac{300}{60} \\
&= 5 \\
m &= \frac{b}{1000} \\
&= \frac{20}{1000} \\
&= \frac{1}{50}\,\text{kg} \\
K &= \frac{mv^2}{2} \\
&= \frac{\frac{1}{50} \times 400^2}{2} \\
&= 1600\,\text{J} \\
P &= rK \\
&= 5 \times 1600 \\
&= 8000\,\text{W}
\end{aligned}
$$

Each bullet carries away the momentum $m\,v$, so the gun is pushed back with an average force equal to the momentum leaving each second:

$$
\begin{aligned}
F &= rmv \\
&= 5 \times \frac{1}{50} \times 400 \\
&= 40\,\text{N}
\end{aligned}
$$

> Result.
>
> The gun gives the bullets 8000 W and must be held with an average force of 40 N.

Question 41. A motor and pump in a chain, and the energy bill

An electric motor drives a pump that lifts 30 kg of water a second through 12 m. The motor turns 80% of the electrical power into mechanical power at its shaft, and the pump turns 75% of that into useful power. Find the electrical power drawn, the overall efficiency, the electrical energy used in 5 hours in kilowatt-hours, and its cost at Rs 8 per kilowatt-hour.

Solution

Work back along the chain from the useful power. Each stage must be supplied with more than it gives out:

$$
\begin{aligned}
P_u &= rgh \\
&= 30 \times 10 \times 12 \\
&= 3600\,\mathrm{W} \\
P_s &= \frac{100 P_u}{e_2} \\
&= \frac{100 \times 3600}{75} \\
&= 4800\,\mathrm{W} \\
P_e &= \frac{100 P_s}{e_1} \\
&= \frac{100 \times 4800}{80} \\
&= 6000\,\mathrm{W} \\
e &= \frac{e_1 e_2}{100} \\
&= \frac{80 \times 75}{100} \\
&= 60
\end{aligned}
$$

A kilowatt-hour is the energy delivered by one kilowatt in one hour, so the energy in those units is the power in kilowatts times the time in hours:

$$
\begin{aligned}
E &= \frac{P_e}{1000} H \\
&= \frac{6000}{1000} \times 5 \\
&= 30\,\mathrm{kWh} \\
C &= Ec \\
&= 30 \times 8 \\
&= 240
\end{aligned}
$$

> Result.
>
> The useful power is 3600 W, the shaft power 4800 W and the electrical power 6000 W. The overall efficiency is 60%, the product of the two. In 5 hours the motor uses 30 kWh, which costs Rs 240.

Question 42. An escalator

An escalator carries 60 people a minute to a floor 5 m higher. Take the average mass of a person as 70 kg; the people stand still on the steps. The drive turns 70% of the power supplied to it into useful power. Find the useful power and the power supplied.

Solution

The useful work is the gain in potential energy of the people carried up. In one minute:

$$
\begin{aligned}
W &= nmgh \\
&= 60 \times 70 \times 10 \times 5 \\
&= 210000\,\mathrm{J} \\
P &= \frac{W}{60} \\
&= \frac{210000}{60} \\
&= 3500\,\mathrm{W} \\
S &= \frac{100P}{e} \\
&= \frac{100 \times 3500}{70} \\
&= 5000\,\mathrm{W}
\end{aligned}
$$

> Result.
>
> The escalator delivers 3500 W of useful power and must be supplied with 5000 W. The speed of the steps does not matter, only how many people reach the top each minute.

Question 43. The power of the heart

A resting heart pumps about 5 litres of blood a minute into the main artery against an average pressure of about 12 kPa. Estimate the useful power of the heart, and the work it does in a day.

Solution

Think of the blood being pushed along a tube of cross-section S at speed v. The force is the pressure times the area, so the power is

$$P = F\,v = (p\,S)\,v = p\,(S\,v)$$

and $S\,v$ is the volume passing each second. Power is pressure times the volume flow rate. In SI units, with 1 litre equal to $10^{-3}\ \mathrm{m}^3$, 1 kPa equal to 1000 Pa and 86400 s in a day:

$$
\begin{aligned}
Q &= \frac{V}{60000} \\
&= \frac{5}{60000} \\
&= \frac{1}{12000}\ \mathrm{m^3/s} \\
p_a &= 1000p \\
&= 1000 \times 12 \\
&= 12000\ \mathrm{Pa} \\
P &= p_a Q \\
&= 12000 \times \frac{1}{12000} \\
&= 1\ \mathrm{W} \\
E &= PD \\
&= 1 \times 86400 \\
&= 86400\ \mathrm{J}
\end{aligned}
$$

> Result.
>
> The heart delivers about 1 W of useful power, and about 86400 J of work in a day.

Question 44. Horsepower

One horsepower is 746 W. A 2 horsepower motor, all of whose power is useful, lifts a load of 373 kg at a steady speed. Find the time it takes to lift the load through 10 m, and the speed of the load.

Solution

At steady speed the work done is the gain in potential energy, and the time is the work divided by the power:

$$
\begin{aligned}
P &= nw \\
&= 2 \times 746 \\
&= 1492 \,\mathrm{W} \\
W &= mgh \\
&= 373 \times 10 \times 10 \\
&= 37300 \,\mathrm{J} \\
T &= \frac{W}{P} \\
&= \frac{37300}{1492} \\
&= 25 \,\mathrm{s} \\
v &= \frac{h}{T} \\
&= \frac{10}{25} \\
&= 0.4 \,\mathrm{m/s}
\end{aligned}
$$

> Result.
>
> The motor delivers 1492 W, and lifts the load in 25 s, at 0.4 m/s.

Power, part 3: questions 45 to 62

Power from a law of motion, connected bodies and machines, two Irodov-style questions, a helicopter, a rocket, fuel and food, and then work and power seen from different frames of reference, including the work and power of a pseudo force. Some questions have numbers and some are worked in symbols. Every number in a solution is worked out by ApasScript. Where it is needed, take the acceleration of free fall as 10 $\mathrm{m/s^2}$.

Question 45. Power from a position-time law

A particle of mass 2 kg moves along the x axis. Its position in metres at the time t seconds is

$$x = t^3 - 6\,t^2 + 9\,t$$

Find the power of the net force on it at 4 s and at 0.5 s. At what moments is the power zero?

Solution

The velocity is the rate of change of position and the acceleration the rate of change of velocity:

$$v = 3\,t^2 - 12\,t + 9 = 3\,(t-1)(t-3)$$

$$a = 6\,t - 12 = 6\,(t-2)$$

The net force is $m\,a$, so the power is $P = m\,a\,v$. At the two moments:

$$\begin{aligned} v_1 &= 3T_1^2 - 12T_1 + 9 \\ &= 3 \times 4^2 - 12 \times 4 + 9 \\ &= 9\,\mathrm{m/s} \\ a_1 &= 6T_1 - 12 \\ &= 6 \times 4 - 12 \\ &= 12\,\mathrm{m/s^2} \\ P_1 &= ma_1v_1 \\ &= 2 \times 12 \times 9 \\ &= 216\,\mathrm{W} \\ v_2 &= 3T_2^2 - 12T_2 + 9 \\ &= 3 \times 0.5^2 - 12 \times 0.5 + 9 \\ &= 3.75\,\mathrm{m/s} \\ a_2 &= 6T_2 - 12 \\ &= 6 \times 0.5 - 12 \\ &= -9\,\mathrm{m/s^2} \\ P_2 &= ma_2v_2 \\ &= 2 \times (-9) \times 3.75 \\ &= -67.5\,\mathrm{W} \end{aligned}$$

The power is zero when either factor is zero: when the particle is momentarily at rest (1 s and 3 s) or when the net force is zero (2 s).

> Result.
>
> The power is 216 W at 4 s and -67.5 W at 0.5 s. It is zero at 1 s, 2 s and 3 s. It is negative whenever the force is against the velocity, that is, while the particle is slowing down: before 1 s and between 2 s and 3 s.

Question 46. Power when the speed grows as the root of the distance

A trolley of mass m starts from rest on a straight track. Its speed depends on the distance s covered as $v = c\sqrt{s}$, where c is a constant. Find the power of the net force as a function of the distance and as a function of time.

Solution

Squaring, $v^2 = c^2\, s$. For motion from rest with a constant acceleration a we have $v^2 = 2\,a\,s$, so here the acceleration is constant and the force is constant:

$$a = \tfrac{1}{2}\,c^2$$

$$F = \tfrac{1}{2}\,m\,c^2$$

The power is the force times the speed. In terms of time the speed is $a\,t$:

> Result.
>
> $$P = \tfrac{1}{2}\,m\,c^3\,\sqrt{s}$$
>
> $$P = \tfrac{1}{4}\,m\,c^4\,t$$

Question 47. Power when the kinetic energy grows as the square of the distance

A body of mass m moves along a straight line. Its kinetic energy depends on the distance s from the start as $K = b\,s^2$, where b is a constant. Find the net force and its power as functions of the distance.

Solution

Over a small step the net force does the work $F\,ds$, and this is the gain in kinetic energy. So the force is the rate at which kinetic energy grows with distance:

$$F = \frac{dK}{ds} = 2\,b\,s$$

The speed follows from the kinetic energy, $\tfrac{1}{2}\,m\,v^2 = b\,s^2$:

$$v = s\,\sqrt{\frac{2\,b}{m}}$$

Result.

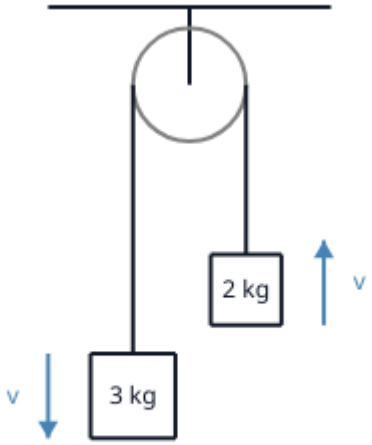
$$F = 2\,b\,s$$

$$P = F\,v = 2\,b\,s^2\,\sqrt{\frac{2\,b}{m}}$$

The force grows in proportion to the distance, and the power as its square.

Question 48. Two blocks over a pulley

Blocks of mass 3 kg and 2 kg hang from the ends of a light string over a light smooth pulley and are released from rest. For the moment 2 s later, find the power of the tension on each block and the power of gravity on each block. What does the total come to?



Solution

The heavier block goes down and the lighter one up, with the same acceleration and the same speed. Adding Newton's second law for the two blocks removes the tension:

$$a = \frac{(m_1 - m_2)\,g}{m_1 + m_2}$$

$$
\begin{aligned}
a &= \frac{(m_1 - m_2)g}{m_1 + m_2} \\
&= \frac{(3-2)\times 10}{3+2} \\
&= 2\,\mathrm{m/s^2} \\
v &= aT \\
&= 2 \times 2 \\
&= 4\,\mathrm{m/s} \\
S &= m_2(g+a) \\
&= 2 \times (10+2) \\
&= 24\,\mathrm{N}
\end{aligned}
$$

Here S is the tension. On the heavier block the tension points up while the block moves down, so its power is negative; on the lighter block it is positive. Gravity is the other way round:

$$
\begin{aligned}
P_1 &= -Sv \\
&= -24 \times 4 \\
&= -96\,\mathrm{W} \\
P_2 &= Sv \\
&= 24 \times 4 \\
&= 96\,\mathrm{W} \\
G_1 &= m_1 g v \\
&= 3 \times 10 \times 4 \\
&= 120\,\mathrm{W} \\
G_2 &= -m_2 g v \\
&= -2 \times 10 \times 4 \\
&= -80\,\mathrm{W} \\
N &= P_1 + P_2 + G_1 + G_2 \\
&= -96 + 96 + 120 + (-80) \\
&= 40\,\mathrm{W}
\end{aligned}
$$

> Result.
>
> The tension is 24 N and the speed 4 m/s. The tension's power is -96 W on the heavier block and 96 W on the lighter one: together zero, so the string only passes energy from one block to the other. Gravity's power is 120 W and -80 W. The total, 40 W, is the rate at which the pair gains kinetic energy.

Question 49. A pulley block: power in and power out

A load of 60 kg is raised at a steady 0.5 m/s by a pulley block in which the load hangs from 4 strands of the rope. (a) For light, smooth pulleys find the pull on the free end, the speed at which the rope must be pulled, and the power put in and delivered. (b) A real block of this kind has an efficiency of 80 %. Find the pull and the power now needed.

Solution

(a) The 4 strands share the weight, so the pull is the weight divided by 4. When the load rises 1 m every strand shortens by 1 m, so the free end must be pulled 4 m: the rope moves 4 times as fast as the load.

$$\begin{aligned} W &= mg \\ &= 60 \times 10 \\ &= 600\,\mathrm{N} \\ F &= \frac{W}{n} \\ &= \frac{600}{4} \\ &= 150\,\mathrm{N} \\ u &= nv \\ &= 4 \times 0.5 \\ &= 2\,\mathrm{m/s} \\ P_i &= Fu \\ &= 150 \times 2 \\ &= 300\,\mathrm{W} \\ P_o &= Wv \\ &= 600 \times 0.5 \\ &= 300\,\mathrm{W} \end{aligned}$$

The machine makes the force smaller and the speed larger by the same factor, so it does not change the power.

(b) With friction, only part of the power put in reaches the load. The speeds are fixed by the rope, so it is the pull that must be larger:

$$\begin{aligned} P_r &= \frac{100 P_o}{e} \\ &= \frac{100 \times 300}{80} \\ &= 375\,\mathrm{W} \\ F_r &= \frac{P_r}{u} \\ &= \frac{375}{2} \\ &= 187.5\,\mathrm{N} \end{aligned}$$

> Result.
>
> The ideal block needs a pull of 150 N at 2 m/s: 300 W in and 300 W out. The real block needs 187.5 N and 375 W.

Question 50. A particle on a circle whose centripetal acceleration grows as the square of time

A particle of mass m moves round a circle of radius R. Its acceleration towards the centre grows with time as $a_n = b\,t^2$, where b is a constant. Find the power of all the forces acting on it at the time t, and the average of that power over the first t seconds.

Solution

The acceleration towards the centre is v^2/R, so the speed is known:

$$v^2 = b\,R\,t^2$$

$$v = t\,\sqrt{b\,R}$$

The part of the net force towards the centre is at right angles to the velocity and delivers no power. Only the part along the circle does, and that part is the mass times the rate of change of the speed, $m\sqrt{b\,R}$. So

$$P = m\sqrt{b\,R}\cdot t\,\sqrt{b\,R}$$

The power grows in proportion to time from zero, so its average over the first t seconds is half its final value.

> Result.
>
> $$P = m\,b\,R\,t$$
>
> $$P_{\text{av}} = \tfrac{1}{2}\,m\,b\,R\,t$$

Question 51. Friction that grows with distance

A small body of mass m slides onto a rough level surface at $x = 0$ with speed u. The coefficient of friction is not the same everywhere: it grows with distance as $k = \alpha\,x$. Find the distance the body travels before it stops, the total work of friction, and the average power of friction over the whole motion.

Solution

The friction force is $k\,m\,g = \alpha\,m\,g\,x$, against the motion. Its work up to the stopping point x_s is the area under the force-distance line, a triangle, and it removes all the kinetic energy:

$$\tfrac{1}{2}\,\alpha\,m\,g\,x_s^2 = \tfrac{1}{2}\,m\,u^2$$

For the average power the time is needed. The equation of motion is

$$\frac{d^2x}{dt^2} = -\,\alpha\, g\, x$$

This is the equation of an oscillation about $x = 0$ with $\omega = \sqrt{\alpha\, g}$. The body starts at the centre with its greatest speed and stops at the extreme, which takes a quarter of a period, $\pi/(2\,\omega)$. The average power is the work divided by this time.

> Result.
>
> $$x_s = \frac{u}{\sqrt{\alpha\, g}}$$
>
> $$W = -\tfrac{1}{2}\, m\, u^2$$
>
> $$P_{\text{av}} = -\,\frac{m\, u^2\, \sqrt{\alpha\, g}}{\pi}$$

Question 52. A helicopter hovering

A helicopter of mass 1200 kg hovers. Take a simple model: its rotor, which sweeps an area of $100\ \text{m}^2$, takes still air and throws it straight down as a jet of that cross-section. The density of air is $1.2\ \text{kg/m}^3$. Find the speed of the jet and the power the engine must give to the air.

Solution

If the jet moves down at speed v, the mass thrown down each second is $\rho\, S\, v$, and it carries the downward momentum $\rho\, S\, v^2$. The rotor pushes the air down with that force and the air pushes the rotor up with the same force, which must equal the weight:

$$\rho\, S\, v^2 = M\, g$$

The engine's power is the kinetic energy given to the air each second:

$$P = \tfrac{1}{2}\, \rho\, S\, v^3 = \tfrac{1}{2}\, M\, g\, v$$

$$
\begin{aligned}
W &= Mg \\
&= 1200 \times 10 \\
&= 12000\,\mathrm{N} \\
v &= \sqrt{\frac{W}{\rho S}} \\
&= \sqrt{\frac{12000}{1.2 \times 100}} \\
&= 10\,\mathrm{m/s} \\
r &= \rho S v \\
&= 1.2 \times 100 \times 10 \\
&= 1200\,\mathrm{kg/s} \\
P &= \frac{r v^2}{2} \\
&= \frac{1200 \times 10^2}{2} \\
&= 60000\,\mathrm{W}
\end{aligned}
$$

> Result.
>
> The air must be thrown down at 10 m/s, 1200 kg of it each second, and the engine gives it 60 kW. The helicopter does not rise, so no work is done on it at all: every watt goes into the air. A larger rotor moves more air more slowly and needs less power for the same weight.

Question 53. A rocket engine: thrust power and the power in the exhaust

A rocket engine throws out 50 kg of gas a second at 2000 m/s relative to the rocket. Find the thrust. Then find the power delivered by the thrust and the kinetic energy carried off by the exhaust each second, as seen from the ground, (a) when the rocket is held at rest on a test stand, and (b) when it is flying at 1000 m/s.

Solution

The thrust is the momentum given to the gas each second, relative to the rocket:

$$
\begin{aligned}
F &= ru \\
&= 50 \times 2000 \\
&= 100000\,\mathrm{N}
\end{aligned}
$$

(a) At rest. The thrust moves nothing, so its power is zero. All the engine's output is kinetic energy of the exhaust:

$$
\begin{aligned}
P_a &= 0F \\
&= 0 \times 100000 \\
&= 0\,\mathrm{W} \\
E_a &= \frac{ru^2}{2} \\
&= \frac{50 \times 2000^2}{2} \\
&= 100000000\,\mathrm{W}
\end{aligned}
$$

(b) In flight. The thrust now acts on a moving rocket. Seen from the ground the gas leaves at $u - v$ backward, and before it was burnt it was moving forward with the rocket at v:

$$
\begin{aligned}
P_b &= Fv \\
&= 100000 \times 1000 \\
&= 100000000\,\mathrm{W} \\
w &= u - v \\
&= 2000 - 1000 \\
&= 1000\,\mathrm{m/s} \\
E_b &= \frac{rw^2}{2} \\
&= \frac{50 \times 1000^2}{2} \\
&= 25000000\,\mathrm{W} \\
E_0 &= \frac{rv^2}{2} \\
&= \frac{50 \times 1000^2}{2} \\
&= 25000000\,\mathrm{W} \\
N &= P_b + E_b - E_0 \\
&= 100000000 + 25000000 - 25000000 \\
&= 100000000\,\mathrm{W}
\end{aligned}
$$

> Result.
>
> The thrust is 100 kN. On the stand its power is 0 W and the exhaust carries off 100 MW. In flight at 1000 m/s the thrust delivers 100 MW to the rocket and the exhaust carries only 25 MW. Counting the kinetic energy the gas already had, the engine's output is 100 MW in both cases: the same engine, but in flight more of its output reaches the rocket.

Question 54. Fuel used by a car

A car cruises at a steady 25 m/s on a level road against a total resistance of 600 N. Its engine turns 25 % of the heat of the fuel into work at the wheels, and a litre of the fuel gives 32 MJ of heat. Find the power at the wheels, the rate at which the fuel supplies heat, and the fuel used in 100 km.

Solution

At steady speed the driving force equals the resistance:

$$\begin{aligned} P &= Rv \\ &= 600 \times 25 \\ &= 15000\ \mathrm{W} \\ P_f &= \frac{100P}{e} \\ &= \frac{100 \times 15000}{25} \\ &= 60000\ \mathrm{W} \end{aligned}$$

Over the journey the work at the wheels is the resistance times the distance. A newton times a kilometre is a kilojoule, so dividing by 1000 gives megajoules:

$$\begin{aligned} W &= \frac{Rd}{1000} \\ &= \frac{600 \times 100}{1000} \\ &= 60\ \mathrm{MJ} \\ E &= \frac{100W}{e} \\ &= \frac{100 \times 60}{25} \\ &= 240\ \mathrm{MJ} \\ L &= \frac{E}{q} \\ &= \frac{240}{32} \\ &= 7.5 \end{aligned}$$

> Result.
>
> The wheels receive 15000 W and the fuel supplies heat at 60000 W. In 100 km the work at the wheels is 60 MJ, which takes 240 MJ of heat: 7.5 litres of fuel. The fuel per kilometre depends on the resistance and the efficiency, not directly on the speed.

Question 55. Human power: stairs and a hill

(a) A person of mass 60 kg runs up stairs through a height of 9 m in 15 s. Find the useful power. (b) The same person climbs a hill 700 m high. Muscles turn about 25% of the energy taken from food into useful work. How much food energy does the climb take, in kilocalories? Take 1 kcal as 4.2 kJ.

Solution

(a) The useful work is the gain in potential energy:

$$
\begin{aligned}
P &= \frac{mgh_1}{T} \\
&= \frac{60 \times 10 \times 9}{15} \\
&= 360\ \mathrm{W}
\end{aligned}
$$

(b) The useful work on the hill, in kilojoules, and the food energy needed to supply it:

$$
\begin{aligned}
W &= \frac{mgh_2}{1000} \\
&= \frac{60 \times 10 \times 700}{1000} \\
&= 420\ \mathrm{kJ} \\
E &= \frac{100W}{e} \\
&= \frac{100 \times 420}{25} \\
&= 1680\ \mathrm{kJ} \\
C &= \frac{E}{c} \\
&= \frac{1680}{4.2} \\
&= 400\ \mathrm{kcal}
\end{aligned}
$$

> Result.
>
> On the stairs the useful power is 360 W, which a person can keep up only for a short time. The hill takes 420 kJ of useful work and so about 1680 kJ, that is 400 kcal, of food energy; the other three quarters becomes heat in the body.

Question 56. The same push seen from a train and from the ground

A train runs along a straight level track at a steady 20 m/s. Inside, a trolley of mass 10 kg stands on the smooth floor. A porter pushes it forward, in the direction of the train's motion, with a steady force of 20 N for 3 s. Find the power of this force at the end of the push and the work it has done, (a) as measured in the train and (b) as measured from the ground. Does the work-energy theorem hold in both?

Solution

The train moves steadily, so both frames are inertial and the force and the acceleration are the same in both. The velocity and the displacement are not.

(a) In the train the trolley starts from rest:

$$
\begin{aligned}
a &= \frac{F}{m} \\
&= \frac{20}{10} \\
&= 2\,\mathrm{m/s^2} \\
w &= aT \\
&= 2 \times 3 \\
&= 6\,\mathrm{m/s} \\
s_t &= \frac{aT^2}{2} \\
&= \frac{2 \times 3^2}{2} \\
&= 9\,\mathrm{m} \\
P_t &= Fw \\
&= 20 \times 6 \\
&= 120\,\mathrm{W} \\
W_t &= Fs_t \\
&= 20 \times 9 \\
&= 180\,\mathrm{J} \\
K_t &= \frac{mw^2}{2} \\
&= \frac{10 \times 6^2}{2} \\
&= 180\,\mathrm{J}
\end{aligned}
$$

(b) From the ground the trolley already had the train's velocity, and the train's motion is added to its displacement:

$$
\begin{aligned}
v &= V + w \\
&= 20 + 6 \\
&= 26\,\mathrm{m/s} \\
s_g &= s_t + VT \\
&= 9 + 20 \times 3 \\
&= 69\,\mathrm{m} \\
P_g &= Fv \\
&= 20 \times 26 \\
&= 520\,\mathrm{W} \\
W_g &= Fs_g \\
&= 20 \times 69 \\
&= 1380\,\mathrm{J} \\
K_g &= \frac{mv^2}{2} - \frac{mV^2}{2} \\
&= \frac{10 \times 26^2}{2} - \frac{10 \times 20^2}{2} \\
&= 1380\,\mathrm{J}
\end{aligned}
$$

> Result.
>
> In the train the power is 120 W and the work 180 J; from the ground they are 520 W and 1380 J. Power and work depend on the frame. The work-energy theorem holds in each: the gain in kinetic energy is 180 J in the train and 1380 J from the ground. The extra work seen from the ground is real: the porter's feet push back on the floor with 20 N, and the engine must do that much more work to keep the train at its speed.

Question 57. Braking, seen from the road and from a car that keeps going

A car of mass 1000 kg moving at 20 m/s brakes with locked wheels; the road exerts a friction force of 5000 N on it until it stops. Find the work done by friction on the car and on the road, and the heat produced, (a) in the frame of the road, and (b) in the frame of a second car that keeps moving at 20 m/s.

Solution

The braking time and the distance along the road:

$$
\begin{aligned}
a &= \frac{f}{m} \\
&= \frac{5000}{1000} \\
&= 5\,\mathrm{m/s^2} \\
T &= \frac{u}{a} \\
&= \frac{20}{5} \\
&= 4\,\mathrm{s} \\
s &= \frac{uT}{2} \\
&= \frac{20 \times 4}{2} \\
&= 40\,\mathrm{m}
\end{aligned}
$$

(a) Road frame. The car moves 40 m against the friction force. The road does not move, so the equal and opposite force on it does no work:

$$
\begin{aligned}
W_c &= -fs \\
&= -5000 \times 40 \\
&= -200000\,\mathrm{J} \\
W_r &= 0f \\
&= 0 \times 5000 \\
&= 0\,\mathrm{J} \\
Q &= -(W_c + W_r) \\
&= -(-200000 + 0) \\
&= 200000\,\mathrm{J}
\end{aligned}
$$

(b) Moving frame. Here the car starts at rest and ends moving backward at 20 m/s; the road moves backward at 20 m/s all the time. Displacements in this frame are those in the road frame minus uT:

$$
\begin{aligned}
s_c &= s - uT \\
&= 40 - 20 \times 4 \\
&= -40\,\mathrm{m} \\
s_r &= 0 - uT \\
&= 0 - 20 \times 4 \\
&= -80\,\mathrm{m} \\
V_c &= -f s_c \\
&= -5000 \times (-40) \\
&= 200000\,\mathrm{J} \\
V_r &= f s_r \\
&= 5000 \times (-80) \\
&= -400000\,\mathrm{J} \\
Q_2 &= -(V_c + V_r) \\
&= -(200000 + (-400000)) \\
&= 200000\,\mathrm{J}
\end{aligned}
$$

> Result.
>
> In the road frame friction does -200000 J on the car and 0 J on the road. In the moving frame it does 200000 J on the car (the car gains kinetic energy there) and -400000 J on the road. The work on each body depends on the frame, but the total, and so the heat, is the same in both: 200000 J and 200000 J. Heat depends only on how far the surfaces slip over each other.

Question 58. A box on an accelerating truck: static friction and the pseudo force

A truck starts from rest with a steady acceleration of $2\ \mathrm{m/s^2}$. A box of mass 50 kg on its floor does not slip. For the moment 5 s after the start, find the power of each horizontal force on the box and the work done on it so far, (a) in the ground frame and (b) in the frame of the truck.

Solution

(a) Ground frame. The only horizontal force on the box is static friction from the floor. It gives the box the truck's acceleration, and it acts on a moving box, so it does work:

$$
\begin{aligned}
f &= mb \\
&= 50 \times 2 \\
&= 100\,\mathrm{N} \\
v &= bT \\
&= 2 \times 5 \\
&= 10\,\mathrm{m/s} \\
s &= \frac{bT^2}{2} \\
&= \frac{2 \times 5^2}{2} \\
&= 25\,\mathrm{m} \\
P_g &= fv \\
&= 100 \times 10 \\
&= 1000\,\mathrm{W} \\
W_g &= fs \\
&= 100 \times 25 \\
&= 2500\,\mathrm{J}
\end{aligned}
$$

Static friction can do work: there is no slipping, but the point where it acts is moving.

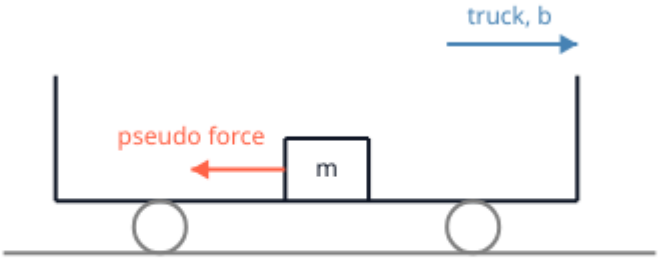
(b) Truck frame. This frame accelerates, so every body in it must be given a pseudo force, $m\,b$ backward. On the box it balances the friction, and the box stays at rest. A force on a body that does not move delivers no power:

$$
\begin{aligned}
F_p &= -mb \\
&= -50 \times 2 \\
&= -100\,\mathrm{N} \\
N &= f + F_p \\
&= 100 + (-100) \\
&= 0\,\mathrm{N} \\
P_t &= 0f \\
&= 0 \times 100 \\
&= 0\,\mathrm{W} \\
P_p &= 0F_p \\
&= 0 \times (-100) \\
&= 0\,\mathrm{W}
\end{aligned}
$$

> Result.
>
> From the ground, friction delivers 1000 W at 5 s and has done 2500 J of work, which is the box's kinetic energy. In the truck the friction of 100 N forward and the pseudo force of -100 N (the minus sign means backward) add up to 0 N; each delivers 0 W and does no work, and the box has no kinetic energy. Both accounts are consistent.

Question 59. A block sliding on the smooth floor of an accelerating truck: work of the pseudo force

A block of mass 5 kg rests on the smooth floor of a truck. The truck starts from rest with a steady acceleration of $2\ \mathrm{m/s^2}$. Working in the frame of the truck, find the power of the pseudo force on the block 3 s later and the work it has done by then. Compare with the ground frame.



Solution

In the truck's frame a pseudo force $m\,b$ acts on the block, pointing backward. The floor is smooth, so nothing balances it, and the block accelerates backward at b relative to the truck. The pseudo force is along this motion:

$$
\begin{aligned}
F &= mb \\
&= 5 \times 2 \\
&= 10\,\mathrm{N} \\
w &= bT \\
&= 2 \times 3 \\
&= 6\,\mathrm{m/s} \\
s &= \frac{bT^2}{2} \\
&= \frac{2 \times 3^2}{2} \\
&= 9\,\mathrm{m} \\
P &= Fw \\
&= 10 \times 6 \\
&= 60\,\mathrm{W} \\
W &= Fs \\
&= 10 \times 9 \\
&= 90\,\mathrm{J} \\
K &= \frac{mw^2}{2} \\
&= \frac{5 \times 6^2}{2} \\
&= 90\,\mathrm{J}
\end{aligned}
$$

> Result.
>
> In the truck's frame the pseudo force of 10 N delivers 60 W at 3 s and has done 90 J of work, which is exactly the block's kinetic energy in that frame, 90 J. So in an accelerating frame the work-energy theorem holds once the work of the pseudo force is counted. From the ground there is no horizontal force on the block at all: it stays where it was while the truck drives out from under it, and no work is done.

Question 60. A ball dropped in a lift that is speeding up

A lift moves upward with a steady acceleration of $2\ \mathrm{m/s^2}$. Inside it a ball of mass 0.5 kg is released from rest, relative to the lift, at a height of 1.5 m above the floor. Working in the frame of the lift, find the speed with which the ball reaches the floor, the work done on it by gravity and by the pseudo force, and the power of each just before it lands.

Solution

In the frame of the lift two forces act on the falling ball, both downward: gravity $m\,g$ and the pseudo force $m\,b$ (opposite to the lift's acceleration). The ball falls relative to the lift with the acceleration $g + b$:

$$
\begin{aligned}
g_e &= g + b \\
&= 10 + 2 \\
&= 12\,\mathrm{m/s^2} \\
w &= \sqrt{2 g_e h} \\
&= \sqrt{2 \times 12 \times 1.5} \\
&= 6\,\mathrm{m/s} \\
T &= \frac{w}{g_e} \\
&= \frac{6}{12} \\
&= 0.5\,\mathrm{s}
\end{aligned}
$$

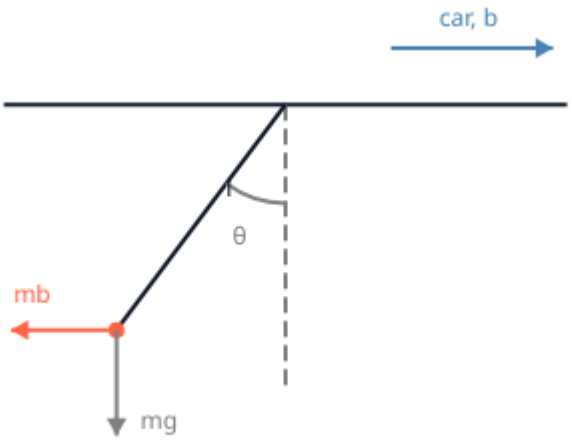
Each force is along the motion and acts through the height 1.5 m:

$$
\begin{aligned}
W_g &= mgh \\
&= 0.5 \times 10 \times 1.5 \\
&= 7.5\,\mathrm{J} \\
W_p &= mbh \\
&= 0.5 \times 2 \times 1.5 \\
&= 1.5\,\mathrm{J} \\
K &= \frac{mw^2}{2} \\
&= \frac{0.5 \times 6^2}{2} \\
&= 9\,\mathrm{J} \\
P_g &= mgw \\
&= 0.5 \times 10 \times 6 \\
&= 30\,\mathrm{W} \\
P_p &= mbw \\
&= 0.5 \times 2 \times 6 \\
&= 6\,\mathrm{W}
\end{aligned}
$$

> Result.
>
> The ball reaches the floor at 6 m/s relative to the lift, after 0.5 s. Gravity does 7.5 J and the pseudo force 1.5 J; together they give the kinetic energy 9 J seen in the lift. Just before landing gravity delivers 30 W and the pseudo force 6 W. Without the pseudo force the energy account in the lift would not balance.

Question 61. A bob hanging in a car that starts to accelerate

A bob hangs at rest on a light string from the roof of a car. The car suddenly starts moving with a steady acceleration b. Working in the frame of the car, find the greatest angle the string makes with the vertical in the swing that follows.



Solution

In the car's frame a steady pseudo force $m\,b$ acts on the bob, pointing backward. The bob starts at rest and is at rest again at its greatest angle, so between those two points the total work on it is zero. The tension is always at right angles to the motion and does no work. That leaves the pseudo force and gravity.

When the string of length l has swung back through the angle θ, the bob has moved backward by $l\sin\theta$ and risen by $l\,(1-\cos\theta)$. So

$$m\,b\,l\sin\theta = m\,g\,l\,(1-\cos\theta)$$

With $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$ and $1-\cos\theta = 2\sin^2\frac{\theta}{2}$ this becomes

> Result.
>
> $$\tan\frac{\theta_{\max}}{2} = \frac{b}{g}$$
>
> The bob would hang at rest at the angle whose tangent is b/g. Starting from the vertical it swings to twice that angle: the pseudo force keeps delivering power on the way out, and the bob overshoots the rest position.

Question 62. A bead on a spinning rod: work of the centrifugal force

A smooth straight rod turns in a horizontal plane about one end at a steady 10 rad/s. A bead of mass 0.1 kg threaded on it is released at 0.3 m from the axis with no speed along the rod. Working in the frame that turns with the rod, find the work done on the bead by the centrifugal force by the time it reaches 0.5 m, its speed along the rod there, and the power of the centrifugal force at that point.

Solution

In the turning frame the rod is at rest and the bead can only slide along it. Three forces act: the push of the rod, sideways; the Coriolis force, also sideways, at right angles to the bead's velocity; and the centrifugal force $m\,\omega^2\,r$, outward along the rod. Only the last does work. It grows with distance, so its work is an area under the force-distance line:

$$W = \int_{r_1}^{r_2} m\,\omega^2\,r\,dr = \tfrac{1}{2}\,m\,\omega^2\left(r_2^2 - r_1^2\right)$$

This work is the bead's kinetic energy in the turning frame:

$$\begin{aligned} W &= \frac{m\omega^2(r_2^2 - r_1^2)}{2} \\ &= \frac{0.1 \times 10^2 \times (0.5^2 - 0.3^2)}{2} \\ &= 0.8\,\mathrm{J} \\ u &= \sqrt{\frac{2W}{m}} \\ &= \sqrt{\frac{2 \times 0.8}{0.1}} \\ &= 4\,\mathrm{m/s} \\ F &= m\omega^2 r_2 \\ &= 0.1 \times 10^2 \times 0.5 \\ &= 5\,\mathrm{N} \\ P &= Fu \\ &= 5 \times 4 \\ &= 20\,\mathrm{W} \end{aligned}$$

> Result.
>
> The centrifugal force does 0.8 J of work. The bead passes 0.5 m sliding outward at 4 m/s, where the centrifugal force is 5 N and delivers 20 W. The Coriolis force, always at right angles to the velocity in this frame, delivers no power, like the magnetic force on a moving charge.