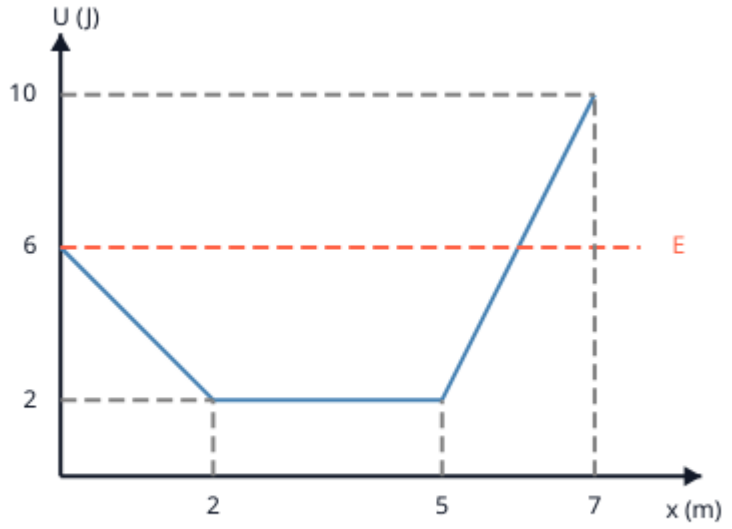
# Energy graphs, equilibrium and stability: questions 31 to 42

Reading a potential-energy graph: the force as its slope, turning points, bound and free motion, and the positions of stable, unstable and neutral equilibrium. Some questions have numbers and some are worked in symbols. Every number in a solution is worked out by ApasScript.

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## Question 31. Reading a potential-energy graph

A particle of mass $0.5$ kg moves along the x axis. Its potential energy is shown in the graph: it falls steadily from $6$ J at $x$ = 0 to $2$ J at $2$ m, stays at $2$ J as far as $5$ m, and then rises steadily to $10$ J at $7$ m. The total energy of the particle is $6$ J. Find the force on it in each of the three regions, its speed in the middle region, and the points where it turns back.




### Solution

**Force.** The force is minus the slope of the graph, $F = -dU/dx$. Where the graph falls to the right the force is to the right; where it is level there is no force; where it rises the force is to the left:

$$
\begin{aligned}
F_1 &= \frac{-(U_b - U_a)}{x_1 - 0} \\
&= \frac{-(2 - 6)}{2 - 0} \\
&= 2\,\mathrm{N} \\
F_2 &= \frac{-(U_b - U_b)}{x_2 - x_1} \\
&= \frac{-(2 - 2)}{5 - 2} \\
&= 0\,\mathrm{N} \\
F_3 &= \frac{-(U_c - U_b)}{x_3 - x_2} \\
&= \frac{-(10 - 2)}{7 - 5} \\
&= -4\,\mathrm{N}
\end{aligned}
$$

**Speed.** The kinetic energy is whatever is left of the total energy after the potential energy, $K = E - U$. In the middle region:

$$
\begin{aligned}
K &= E - U_b \\
&= 6 - 2 \\
&= 4\,\mathrm{J} \\
v &= \sqrt{\frac{2K}{m}} \\
&= \sqrt{\frac{2 \times 4}{0.5}} \\
&= 4\,\mathrm{m/s}
\end{aligned}
$$

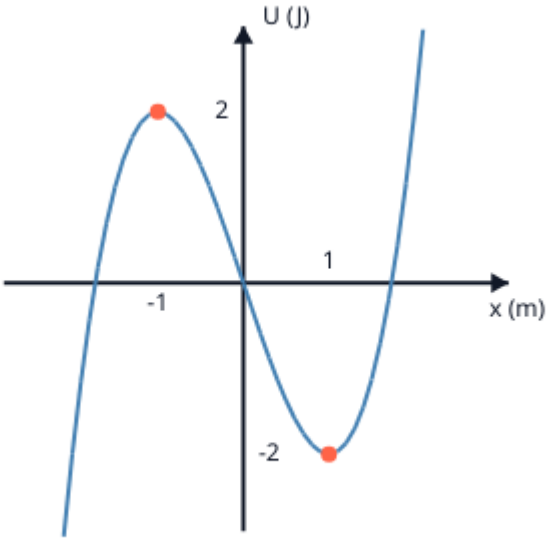
**Turning points.** The particle stops and turns back where the potential energy has risen to the total energy, so that no kinetic energy is left. On the left that is $x$ = 0, where $U$ is $6$ J. On the right the graph climbs steadily from $2$ J at $5$ m to $10$ J at $7$ m, so it reaches $6$ J part of the way up:

$$
\begin{aligned}
x_t &= x_2 + \frac{(E - U_b)(x_3 - x_2)}{U_c - U_b} \\
&= 5 + \frac{(6 - 2) \times (7 - 5)}{10 - 2} \\
&= 6\,\mathrm{m}
\end{aligned}
$$

> Result.
>
> The force is $2$ N in the first region, $0$ N in the second and $-4$ N in the third (the minus sign means towards smaller $x$). In the middle region the kinetic energy is $4$ J and the speed $4$ m/s. The particle moves back and forth between $x$ = 0 and $x$ = $6$ m; it can never be found beyond them, because the kinetic energy would have to be negative there.

## Question 32. Positions of equilibrium and their stability

The potential energy of a particle on the x axis is $U = (x^3 - 3x)$ J, with $x$ in metres, as in the graph. Find the positions of equilibrium and say whether each is stable. Find also the force on the particle at $x$ = 2 m.




**Solution**

The particle is in equilibrium where the force is zero, that is where the graph is level:

$$F = -\frac{dU}{dx} = -3x^2 + 3 = -3(x-1)(x+1)$$

So the equilibrium positions are $x$ = 1 m and $x$ = −1 m. To test each, move the particle a little and see which way the force pushes. The quick test is the sign of the second slope, $d^2U/dx^2 = 6x$:

**At 1 m** the second slope is positive: the graph is a valley. A small push either way brings a force back towards the valley floor. The equilibrium is stable.

**At −1 m** the second slope is negative: the graph is a hilltop. A small push either way brings a force away from the top. The equilibrium is unstable.

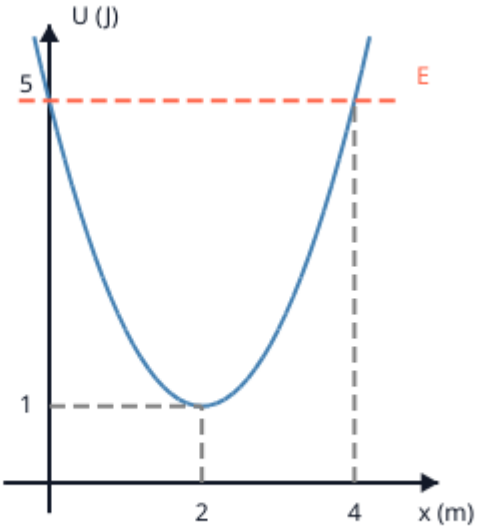
$$\begin{aligned} F &= -3x_1^2 + 3 \\ &= -3 \times 2^2 + 3 \\ &= -9\,\mathrm{N} \\ U_s &= 1^3 - 3 \times 1 \\ &= -2\,\mathrm{J} \\ U_u &= (-1)^3 - 3 \times (-1) \\ &= 2\,\mathrm{J} \end{aligned}$$

> Result.
>
> There is stable equilibrium at $x$ = 1 m (the minimum, $-2$ J) and unstable equilibrium at $x$ = $-1$ m (the maximum, $2$ J). At $2$ m the force is $-9$ N, pushing the particle back towards the minimum.

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## Question 33. A particle in a potential well

A particle of mass $2$ kg moves along the x axis with the potential energy $U = (x^2 - 4x + 5)$ J, with $x$ in metres. Its total energy is $5$ J. Find the equilibrium position, the two turning points, the greatest speed of the particle, and its speed at $x$ = $1$ m.




**Solution**

Completing the square shows the shape of the well at once:

$$U = (x - 2)^2 + 1$$

The lowest point, and so the position of stable equilibrium, is $x$ = 2 m, where $U$ is 1 J. The turning points are where $U = E$, that is $(x - 2)^2 = E - 1$:

$$
\begin{aligned}
U_0 &= 1 \\
&= 1\,\mathrm{J} \\
w &= \sqrt{E - U_0} \\
&= \sqrt{5 - 1} \\
&= 2\,\mathrm{m} \\
x_a &= 2 - w \\
&= 2 - 2 \\
&= 0\,\mathrm{m} \\
x_b &= 2 + w \\
&= 2 + 2 \\
&= 4\,\mathrm{m}
\end{aligned}
$$

The kinetic energy $E - U$ is greatest at the bottom of the well:

$$
\begin{aligned}
K_m &= E - U_0 \\
&= 5 - 1 \\
&= 4\,\mathrm{J} \\
v_m &= \sqrt{\frac{2K_m}{m}} \\
&= \sqrt{\frac{2 \times 4}{2}} \\
&= 2\,\mathrm{m/s} \\
U_1 &= x_1^2 - 4x_1 + 5 \\
&= 1^2 - 4 \times 1 + 5 \\
&= 2\,\mathrm{J} \\
K_1 &= E - U_1 \\
&= 5 - 2 \\
&= 3\,\mathrm{J}
\end{aligned}
$$

> Result.
>
> The particle oscillates about $x$ = 2 m between the turning points $0$ m and $4$ m. Its greatest speed, at the bottom of the well, is $2$ m/s. At $1$ m the potential energy is $2$ J, the kinetic energy $3$ J and the speed $\sqrt{3}$ m/s, about $1.732051$ m/s.

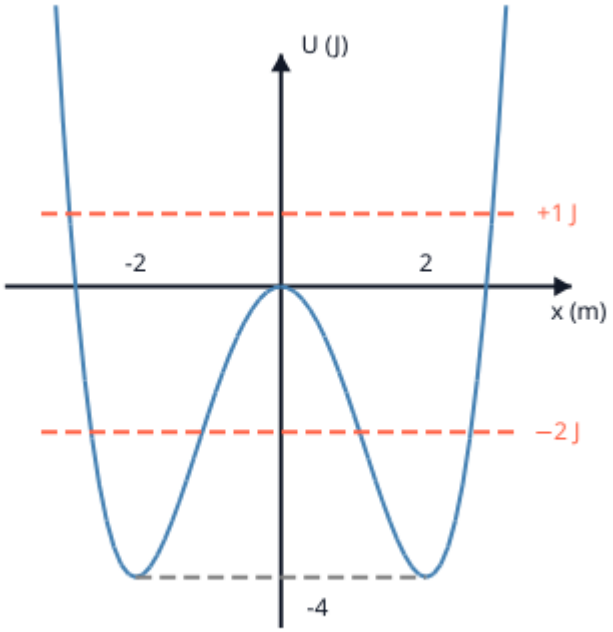
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## Question 34. Two wells and a barrier

A particle of mass $2$ kg on the x axis has the potential energy

$$U = \left(\tfrac{1}{4}\,x^4 - 2\,x^2\right)\ \mathrm{J}$$

with $x$ in metres, as in the graph. Find the equilibrium positions and their stability. A particle sits at rest at the bottom of the right-hand well. What least speed must it be given to cross to the other well? Describe the motion for a total energy of $-2$ J and of +1 J.




**Solution**

The force is minus the slope:

$$F = -\frac{dU}{dx} = -x^3 + 4x = -x\,(x-2)(x+2)$$

It is zero at $x$ = −2 m, 0 and 2 m. The second slope is $3x^2 - 4$: positive at ±2 m (valleys, stable) and negative at 0 (a hilltop, unstable).

$$\begin{aligned} U_w &= \frac{2^4}{4} - 2 \times 2^2 \\ &= -4\,\mathrm{J} \\ U_h &= 0 \\ &= 0\,\mathrm{J} \\ B &= U_h - U_w \\ &= 0 - (-4) \\ &= 4\,\mathrm{J} \\ v &= \sqrt{\frac{2B}{m}} \\ &= \sqrt{\frac{2 \times 4}{2}} \\ &= 2\,\mathrm{m/s} \end{aligned}$$

To cross, the particle must just reach the top of the barrier at $x$ = 0, so its kinetic energy at the bottom of the well must equal the height of the barrier above the well floor.

Result.

There is stable equilibrium at $x$ = 2 m and at $x$ = −2 m, where $U$ is $-4$ J, and unstable equilibrium at $x$ = 0, where $U$ is $0$ J. The barrier is $4$ J high, so the particle needs at least $2$ m/s. With a total energy of −2 J the particle is trapped in one well and oscillates there: the barrier top is above its energy. With +1 J it passes over the barrier and oscillates across both wells, slowing as it crosses the middle.

---

## Question 35. Attraction far away, repulsion close in

The potential energy of a particle at a distance $r$ from a fixed centre is

$$U(r) = \frac{a}{r^2} - \frac{b}{r}$$

where $a$ and $b$ are positive constants. Find the distance at which the particle is in equilibrium, and show that the equilibrium is stable. Find also the greatest attractive force.

**Solution**

The force along the outward direction is minus the slope:

$$F = -\frac{dU}{dr} = \frac{2\,a}{r^3} - \frac{b}{r^2}$$

It is zero where $2a/r^3 = b/r^2$. Closer in, the first term wins and the force is outward (repulsion); farther out, the second wins and the force is inward (attraction). Either way the particle is pushed back to the equilibrium distance, so the equilibrium is stable: it is the bottom of a potential well.

The attraction is zero at the equilibrium distance and dies away again far off, so it is greatest somewhere between. Setting the slope of $F$ to zero:

$$\frac{dF}{dr} = -\frac{6\,a}{r^4} + \frac{2\,b}{r^3} = 0$$

Result.

$$r_0 = \frac{2\,a}{b}$$

$$U(r_0) = -\frac{b^2}{4\,a}$$

$$F_{\max} = \frac{b^3}{27\,a^2}$$

The greatest attraction occurs at $r = 3a/b$, one and a half times the equilibrium distance.

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## Question 36. Two atoms: the 6-12 potential

The potential energy of two atoms a distance $r$ apart is often written as

$$U(r) = \frac{A}{r^{12}} - \frac{B}{r^{6}}$$

where $A$ and $B$ are positive constants. Find the separation at which the atoms are in equilibrium and the energy needed to pull them completely apart from there.

**Solution**

At equilibrium the slope of the potential energy is zero:

$$\frac{dU}{dr} = -\frac{12\,A}{r^{13}} + \frac{6\,B}{r^{7}} = 0$$

which gives $r^6 = 2A/B$. Putting this back, $A/r^{12} = B^2/(4A)$ and $B/r^6 = B^2/(2A)$. Far apart the potential energy is zero, so the energy needed to separate the atoms is the depth of the well.

> Result.
>
> $$r_0 = \left(\frac{2\,A}{B}\right)^{1/6}$$
>
> $$U(r_0) = -\frac{B^2}{4\,A}$$
>
> The energy needed to pull the atoms apart is $B^2/(4A)$.

---

## Question 37. Small oscillations about a stable equilibrium

A particle of mass $1.5$ kg has the potential energy $U = (x^3 - 3x)$ J of Question 32 and rests at the stable equilibrium position, $x$ = 1 m. It is given a small push. Find the angular frequency and the period of the small oscillations that follow.

**Solution**

Near the bottom, any smooth well looks like a parabola. Writing $s$ for the small displacement from the minimum, the potential energy is very nearly $U_{\min} + \tfrac{1}{2}\,k\,s^2$, where $k$ is the second slope of the potential energy at the minimum. That is the potential energy of a spring of stiffness $k$, so the particle oscillates as if on such a spring, with $\omega = \sqrt{k/m}$. Here the second slope is $6x$, taken at $x$ = 1 m:

$$\begin{aligned} k &= 6 \times 1 \\ &= 6\,\mathrm{N/m} \\ w &= \sqrt{\frac{k}{m}} \\ &= \sqrt{\frac{6}{1.5}} \\ &= 2\,\mathrm{rad/s} \\ T &= \frac{2\pi}{w} \\ &= \frac{2\pi}{2} \\ &= 3.141592654\,\mathrm{s} \end{aligned}$$

> Result.
>
> The effective stiffness is $6$ N/m, the angular frequency $2$ rad/s and the period $\pi$ s, about $3.141593$ s. This holds only for small oscillations: farther from the minimum the well is not a parabola.

---

## Question 38. Stable, unstable or neutral?

A particle on the x axis can have one of these four potential energies, with $c$ a positive constant. In each case $x = 0$ is a position of equilibrium. Decide whether that equilibrium is stable, unstable or neutral.

(i) $U = c\,x^2$

(ii) $U = -\,c\,x^2$

(iii) $U = c$, the same everywhere

(iv) $U = c\,x^4$

**Solution**

Displace the particle slightly and ask which way the force $F = -dU/dx$ pushes.

**First.** $F = -2c\,x$: opposite to the displacement, so the particle is pushed back. Stable; the graph is a valley.

**Second.** $F = +2c\,x$: along the displacement, so the particle is pushed farther away. Unstable; the graph is a hilltop.

**Third.** $F = 0$ everywhere: the particle stays wherever it is put. Neutral; the graph is level, like a ball on a flat table.

**Fourth.** $F = -4c\,x^3$: opposite to the displacement again, so the equilibrium is stable. But here the second slope, $12c\,x^2$, is zero at the equilibrium point. So a zero second slope does not settle the matter by itself: one must look at the shape of the graph, or at the direction of the force, on each side.

> Result.
>
> Stable, unstable, neutral and stable. The rule: a minimum of potential energy is stable, a maximum is unstable, and a level stretch is neutral. A positive second slope proves a minimum, and a negative one a maximum, but a zero second slope proves nothing.

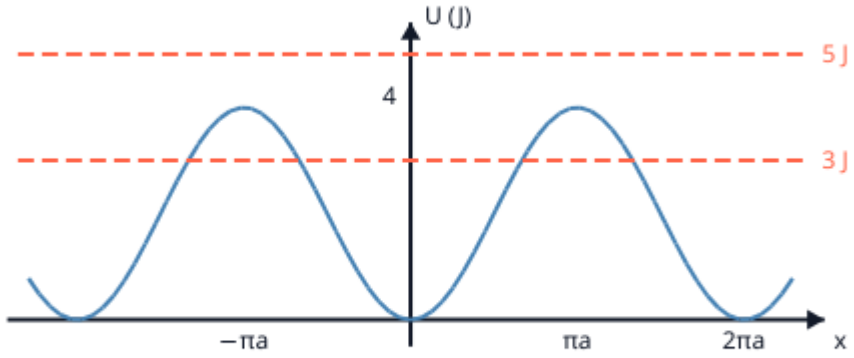
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## Question 39. A hilly road: trapped or running free

A particle moves along the x axis with the potential energy

$$U = A\left(1 - \cos\frac{x}{a}\right)$$

a row of equal valleys and hills, with the constant $A$ equal to $2$ J and $a$ a constant length. Find the positions of stable and unstable equilibrium. Describe the motion when the total energy is $3$ J and when it is $5$ J, and for the second case find the least and greatest kinetic energy.




**Solution**

The valleys are where the cosine is +1 and $U$ is zero: $x = 0, \pm 2\pi a, \ldots$ These are positions of stable equilibrium. The hilltops are where the cosine is $-1$ and $U$ is $2A$: $x = \pm\pi a, \pm 3\pi a, \ldots$ These are positions of unstable equilibrium.

$$\begin{aligned} U_t &= 2U_0 \\ &= 2 \times 2 \\ &= 4\,\text{J} \end{aligned}$$

**Trapped.** A total energy of $3$ J is below the hilltops, so the particle is trapped in one valley. It turns back where $U$ equals its energy:

$$\begin{aligned} c &= 1 - \frac{E_1}{U_0} \\ &= 1 - \frac{3}{2} \\ &= -\frac{1}{2} \\ K_a &= E_1 - 0 \\ &= 3 - 0 \\ &= 3\,\text{J} \end{aligned}$$

The cosine of $x/a$ at the turning points is $-\frac{1}{2}$, so $x/a = \pm 2\pi/3$: the particle swings two thirds of the way to the hilltops on each side.

**Free.** A total energy of $5$ J is above the hilltops, so nothing stops the particle. It runs on in one direction for ever, fastest in the valleys and slowest over the hilltops:

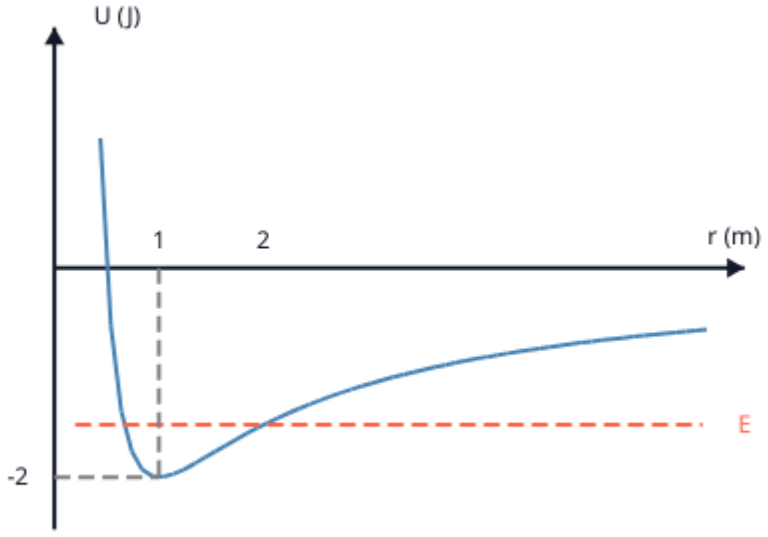
$$
\begin{aligned}
K_1 &= E_2 - 0 \\
&= 5 - 0 \\
&= 5\ \mathrm{J} \\
K_2 &= E_2 - U_t \\
&= 5 - 4 \\
&= 1\ \mathrm{J}
\end{aligned}
$$

> Result.
>
> The hilltops are at $4$ J. With $3$ J the particle oscillates in one valley between $x = -2\pi a/3$ and $x = +2\pi a/3$, with at most $3$ J of kinetic energy. With $5$ J it is free: its kinetic energy varies between $1$ J over the hilltops and $5$ J in the valleys. A pendulum behaves in just this way: it swings if its energy is too small to carry it over the top, and whirls round if it is large enough.

---

## Question 40. Bound and free in an attraction-repulsion well

A particle has the potential energy of Question 35, $U = a/r^2 - b/r$, with $a$ = 2 J m$^2$ and $b$ = 4 J m, as in the graph. (a) Find the equilibrium distance and the depth of the well. (b) The particle has a total energy of $-1.5$ J. Between what distances does it move? (c) What least energy must be given to a particle at rest at the bottom of the well to free it?




**Solution**

**(a)** From Question 35 the equilibrium distance is $2a/b$ and the potential energy there is $-b^2/(4a)$:

$$
\begin{aligned}
r_0 &= \frac{2a}{b} \\
&= \frac{2 \times 2}{4} \\
&= 1\,\mathrm{m} \\
U_0 &= \frac{-b^2}{4a} \\
&= \frac{-4^2}{4 \times 2} \\
&= -2\,\mathrm{J}
\end{aligned}
$$

**(b)** The energy is negative, so the particle cannot get far away, where $U$ is zero: it is bound. It turns back where $U = E$. Multiplying $a/r^2 - b/r = E$ by $r^2$ gives the quadratic $E\,r^2 + b\,r - a = 0$, with the roots

$$
\begin{aligned}
D &= b^2 + 4Ea \\
&= 4^2 + 4 \times \frac{-3}{2} \times 2 \\
&= 4 \\
r_1 &= \frac{-b + \sqrt{D}}{2E} \\
&= \frac{-4 + \sqrt{4}}{2 \times \frac{-3}{2}} \\
&= \frac{2}{3}\,\mathrm{m} \\
r_2 &= \frac{-b - \sqrt{D}}{2E} \\
&= \frac{-4 - \sqrt{4}}{2 \times \frac{-3}{2}} \\
&= 2\,\mathrm{m}
\end{aligned}
$$

**(c)** A free particle needs a total energy of at least zero. Starting at rest at the bottom of the well:

$$
\begin{aligned}
B &= 0 - U_0 \\
&= 0 - (-2) \\
&= 2\,\mathrm{J}
\end{aligned}
$$

> Result.
>
> The equilibrium distance is $1$ m and the well is $2$ J deep ($U$ = $-2$ J at the bottom). With a total energy of $-1.5$ J the particle moves in and out between $\frac{2}{3}$ m and $2$ m. At least $2$ J must be given to free it. A particle whose total energy is positive comes in from far away, turns back once near the centre and never returns.

## Question 41. Floating above a repelling floor

A small magnet of mass $0.2$ kg can move only up and down above a repelling base. At a height $y$ above the base the repulsion gives it the potential energy $c/y$, with $c$ equal to $0.5$ J m, in addition to its gravitational potential energy. Find the height at which it floats in equilibrium, show that the equilibrium is stable, and find the stiffness of the equivalent spring for small up-and-down oscillations.

**Solution**

The total potential energy is

$$U(y) = \frac{c}{y} + m\,g\,y$$

The first term falls with height and the second rises, so the sum has a lowest point. There the slope is zero:

$$\frac{dU}{dy} = -\frac{c}{y^2} + m\,g = 0$$

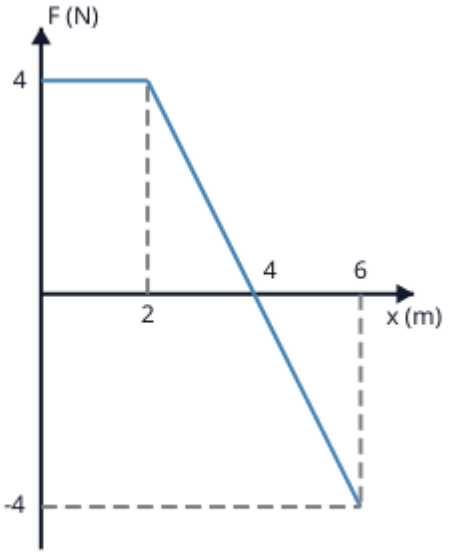
The second slope, $2c/y^3$, is positive at every height. So the equilibrium is a minimum and is stable, and the second slope there is the stiffness of the equivalent spring (Question 37):

$$\begin{aligned} W &= mg \\ &= 0.2 \times 10 \\ &= 2\,\mathrm{N} \\ y_0 &= \sqrt{\frac{c}{W}} \\ &= \sqrt{\frac{0.5}{2}} \\ &= 0.5\,\mathrm{m} \\ k &= \frac{2c}{y_0^3} \\ &= \frac{2 \times 0.5}{0.5^3} \\ &= 8\,\mathrm{N/m} \\ U_0 &= \frac{c}{y_0} + W y_0 \\ &= \frac{0.5}{0.5} + 2 \times 0.5 \\ &= 2\,\mathrm{J} \end{aligned}$$

> Result.
>
> The magnet floats $0.5$ m above the base, where the total potential energy has its least value, $2$ J. Pushed down, it is repelled more strongly than gravity pulls; lifted, less strongly: it is pushed back either way. For small oscillations it behaves like a body on a spring of stiffness $8$ N/m.

## Question 42. From a force graph to potential energy

The force on a particle of mass $1$ kg moving along the x axis is shown in the graph: a steady $4$ N from $x$ = 0 to $2$ m, then falling steadily, through zero at $4$ m, to $-4$ N at $6$ m. The particle starts from rest at $x$ = 0. Taking the potential energy as zero at $x$ = 0, find it at $2$ m, $4$ m and $6$ m. Where is the equilibrium position, and is it stable? Find the greatest speed of the particle and its speed at $6$ m.




**Solution**

The potential energy at a point is minus the work done by the force in reaching it from the zero point, that is, minus the area under the force graph, counting area below the axis as negative:

$$U(x) = -\int_0^x F\,dx$$

The areas are a rectangle, a triangle above the axis and an equal triangle below it:

$$
\begin{aligned}
A_1 &= F_0 x_1 \\
&= 4 \times 2 \\
&= 8\,\mathrm{J} \\
A_2 &= \frac{F_0(x_2 - x_1)}{2} \\
&= \frac{4 \times (4 - 2)}{2} \\
&= 4\,\mathrm{J} \\
A_3 &= \frac{-F_0(x_3 - x_2)}{2} \\
&= \frac{-4 \times (6 - 4)}{2} \\
&= -4\,\mathrm{J} \\
U_1 &= -A_1 \\
&= -8 \\
&= -8\,\mathrm{J} \\
U_2 &= -(A_1 + A_2) \\
&= -(8 + 4) \\
&= -12\,\mathrm{J} \\
U_3 &= -(A_1 + A_2 + A_3) \\
&= -(8 + 4 + (-4)) \\
&= -8\,\mathrm{J}
\end{aligned}
$$

The force is zero at $4$ m, so that is the equilibrium position. Just before it the force is forward and just after it is backward: the particle is pushed back from either side, so the equilibrium is stable, and the potential energy is least there. The particle started from rest with $U$ = 0, so its total energy is zero and its kinetic energy is $-U$:

$$
\begin{aligned}
K_m &= 0 - U_2 \\
&= 0 - (-12) \\
&= 12\,\mathrm{J} \\
v_m &= \sqrt{\frac{2K_m}{m}} \\
&= \sqrt{\frac{2 \times 12}{1}} \\
&= 2\sqrt{6}\,\mathrm{m/s} \\
K_3 &= 0 - U_3 \\
&= 0 - (-8) \\
&= 8\,\mathrm{J} \\
v_3 &= \sqrt{\frac{2K_3}{m}} \\
&= \sqrt{\frac{2 \times 8}{1}} \\
&= 4\,\mathrm{m/s}
\end{aligned}
$$

> Result.
>
> The potential energy is $-8$ J at $2$ m, $-12$ J at $4$ m and $-8$ J at $6$ m. The equilibrium at $4$ m is stable. The particle is fastest there, with $12$ J of kinetic energy, about $4.898979$ m/s, and passes $6$ m with $8$ J, at $4$ m/s.