Conservative force and potential energy: questions 1 to 13

When a force is conservative, how to find its potential energy, and how to get the force back from the potential energy. Some questions have numbers and some are worked in symbols. Every number in a solution is worked out by ApasScript. Where it is needed, take the acceleration of free fall as 10 $\mathrm{m/s^2}$.

Question 1. Work that does not depend on the path

A particle moves in the xy plane under the force

$$F = \left(2xy\,\hat{i} + x^2\,\hat{j}\right)\ \mathrm{N}$$

with x and y in metres. Find the work done by the force when the particle goes from A$(1,\ 1)$ to B$(3,\ 2)$ (a) along the x direction first and then along the y direction, and (b) along the y direction first and then along the x direction. What does the result suggest?

Solution

Over a small step the work is $dW = 2xy\,dx + x^2\,dy$. On a leg along the x direction $dy = 0$ and y is fixed; on a leg along the y direction $dx = 0$ and x is fixed.

(a) First along x with y at its starting value, then along y with x at its final value:

$$\begin{aligned} A_1 &= y_1(x_2^2 - x_1^2) \\ &= 1 \times (3^2 - 1^2) \\ &= 8\ \mathrm{J} \\ A_2 &= x_2^2(y_2 - y_1) \\ &= 3^2 \times (2 - 1) \\ &= 9\ \mathrm{J} \\ W_a &= A_1 + A_2 \\ &= 8 + 9 \\ &= 17\ \mathrm{J} \end{aligned}$$

(b) First along y with x at its starting value, then along x with y at its final value:

$$\begin{aligned} B_1 &= x_1^2(y_2 - y_1) \\ &= 1^2 \times (2 - 1) \\ &= 1\ \mathrm{J} \\ B_2 &= y_2(x_2^2 - x_1^2) \\ &= 2 \times (3^2 - 1^2) \\ &= 16\ \mathrm{J} \\ W_b &= B_1 + B_2 \\ &= 1 + 16 \\ &= 17\ \mathrm{J} \end{aligned}$$

> Result.
>
> The work is 17 J by the first route and 17 J by the second: the same. This is what a conservative force does. Here $2xy\,dx + x^2\,dy$ is exactly the change of x^2y, so the work between any two points is the change of x^2y, whatever the path.

Question 2. A test for a conservative force

For each of these forces in the xy plane decide whether it is conservative, and if it is, find its potential energy, taking it as zero at the origin. Here a is a constant.

$$F_1 = a\left(2xy\,\hat{i} + x^2\,\hat{j}\right)$$

$$F_2 = a\left(y^2\,\hat{i} + x^2\,\hat{j}\right)$$

Solution

A force in the plane is conservative when the work $F_x\,dx + F_y\,dy$ is the exact change of some function of position. The test is that the two cross slopes agree:

$$\frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x}$$

First force. $\partial(2xy)/\partial y = 2x$ and $\partial(x^2)/\partial x = 2x$. They agree, so the force is conservative. Its work over a small step is $a\,d(x^2y)$, and potential energy is defined so that its fall equals the work done:

$$dU = -\,dW$$

Second force. $\partial(y^2)/\partial y = 2y$ but $\partial(x^2)/\partial x = 2x$. These differ (except on the line $y = x$), so the work depends on the path and no potential energy exists.

> Result.
>
> $$U_1 = -\,a\,x^2\,y$$
>
> The first force is conservative. The second is not, and has no potential energy.

Question 3. Force from a potential energy

The potential energy of a particle that moves along the x axis is

$$U = \left(3x^2 - 2x^3\right)\ \text{J}$$

with x in metres. Find the force on the particle at x = 2 m and at x = -1 m, and the positions where the force is zero.

Solution

The force is minus the slope of the potential energy: it pushes the particle towards lower potential energy.

$$F = -\frac{dU}{dx} = -6x + 6x^2 = 6x\,(x-1)$$

$$\begin{aligned} F_1 &= 6x_1(x_1 - 1) \\ &= 6 \times 2 \times (2-1) \\ &= 12\ \mathrm{N} \\ F_2 &= 6x_2(x_2 - 1) \\ &= 6 \times (-1) \times (-1-1) \\ &= 12\ \mathrm{N} \end{aligned}$$

> Result.
>
> The force is 12 N at 2 m and 12 N at -1 m, both in the positive x direction. It is zero at x = 0 and at x = 1 m, where the potential-energy curve is level.

Question 4. Force from a potential energy in two dimensions

A particle in the xy plane has the potential energy $U = k\,x\,y$, with k equal to $2\ \mathrm{J/m^2}$. Find the force on it at the point $(3,\ 4)$, with coordinates in metres, and the magnitude of that force.

Solution

Each component of the force is minus the slope of the potential energy in that direction, with the other coordinate held fixed:

$$F_x = -\frac{\partial U}{\partial x} = -k\,y$$

$$F_y = -\frac{\partial U}{\partial y} = -k\,x$$

$$\begin{aligned} F_x &= -ky_1 \\ &= -2 \times 4 \\ &= -8\ \mathrm{N} \\ F_y &= -kx_1 \\ &= -2 \times 3 \\ &= -6\ \mathrm{N} \\ F &= \sqrt{F_x^2 + F_y^2} \\ &= \sqrt{(-8)^2 + (-6)^2} \\ &= 10\ \mathrm{N} \end{aligned}$$

> Result.
>
> The force has components -8 N and -6 N, and its magnitude is 10 N.

Question 5. Potential energy from a force law

A particle moves along the x axis under the force

$$F = -k\,x - b\,x^3$$

where k and b are positive constants. Find its potential energy, taking it as zero at $x = 0$.

Solution

The potential energy at x is minus the work done by the force when the particle is taken there from the reference point:

$$U(x) = -\int_0^x F\,dx = \int_0^x \left(k\,x + b\,x^3\right) dx$$

> Result.
>
> $$U = \tfrac{1}{2}\,k\,x^2 + \tfrac{1}{4}\,b\,x^4$$
>
> The first term is the potential energy of an ordinary spring. The second makes the "spring" stiffer the more it is stretched.

Question 6. A spring: potential energy and the work it does

A spring of stiffness 200 N/m is already stretched by 0.1 m. It is stretched further, to 0.3 m. Find the change in its potential energy and the work done by the spring force. What work must the hand do if the stretching is slow?

Solution

The potential energy of a spring stretched by x is $\frac{1}{2}kx^2$:

$$
\begin{aligned}
U_1 &= \frac{k x_1^2}{2} \\
&= \frac{200 \times 0.1^2}{2} \\
&= 1\ \text{J} \\
U_2 &= \frac{k x_2^2}{2} \\
&= \frac{200 \times 0.3^2}{2} \\
&= 9\ \text{J} \\
D &= U_2 - U_1 \\
&= 9 - 1 \\
&= 8\ \text{J} \\
W_s &= -D \\
&= -8 \\
&= -8\ \text{J} \\
W_h &= D \\
&= 8 \\
&= 8\ \text{J}
\end{aligned}
$$

> Result.
>
> The potential energy rises by 8 J. The spring force does -8 J of work: negative, because it pulls back while the end moves out. A hand stretching it slowly does 8 J. Note that the answer is not $\frac{1}{2}k$ times the square of the extra stretch: the energy goes as the square of the total stretch.

Question 7. The zero of potential energy does not matter

A book of mass 2 kg is moved from a shelf 5 m above the floor to a table 2 m above the floor. The floor itself is 3 m above the ground outside. Find the change in the book's gravitational potential energy, taking the zero of potential energy (a) at the floor and (b) at the ground outside.

Solution

Near the earth's surface the gravitational potential energy is $m\,g$ times the height above the chosen zero level.

(a) Zero at the floor:

$$
\begin{aligned}
U_1 &= mgh_1 \\
&= 2 \times 10 \times 5 \\
&= 100\,\mathrm{J} \\
U_2 &= mgh_2 \\
&= 2 \times 10 \times 2 \\
&= 40\,\mathrm{J} \\
D_a &= U_2 - U_1 \\
&= 40 - 100 \\
&= -60\,\mathrm{J}
\end{aligned}
$$

(b) Zero at the ground outside. Every height is larger by the same amount:

$$
\begin{aligned}
V_1 &= mg(h_1 + d) \\
&= 2 \times 10 \times (5 + 3) \\
&= 160\,\mathrm{J} \\
V_2 &= mg(h_2 + d) \\
&= 2 \times 10 \times (2 + 3) \\
&= 100\,\mathrm{J} \\
D_b &= V_2 - V_1 \\
&= 100 - 160 \\
&= -60\,\mathrm{J}
\end{aligned}
$$

> Result.
>
> With the zero at the floor the potential energy goes from 100 J to 40 J; with the zero at the ground, from 160 J to 100 J. The change is -60 J in both cases. Only changes of potential energy have a physical meaning, so the zero can be put wherever it is convenient.

Question 8. An inverse-square force

A particle is attracted towards a fixed point O with a force that falls off as the square of the distance, $F = A/r^2$, with A equal to $12\ \mathrm{N\,m^2}$. Find its potential energy as a function of the distance r, taking it as zero very far away. Then find the work done by the force when the particle moves from 2 m to 6 m from O.

Solution

The force points inward, so along the outward direction it is $-A/r^2$. The potential energy at r is minus the work done by the force in bringing the particle in from far away:

$$
U(r) = -\int_{\infty}^{r} \left(-\frac{A}{r^2}\right) dr = -\frac{A}{r}
$$

It is negative everywhere: the particle is bound, and work must be done on it to take it far away. The work done by the force between two distances is the fall in potential energy:

$$
\begin{aligned}
U_1 &= \frac{-A}{r_1} \\
&= \frac{-12}{2} \\
&= -6\,\mathrm{J} \\
U_2 &= \frac{-A}{r_2} \\
&= \frac{-12}{6} \\
&= -2\,\mathrm{J} \\
W &= U_1 - U_2 \\
&= -6 - (-2) \\
&= -4\,\mathrm{J}
\end{aligned}
$$

> Result.
>
> The potential energy is $-A/r$: -6 J at 2 m and -2 J at 6 m. The force does -4 J of work on the way out: negative, because it pulls inward while the particle moves outward.

Question 9. Friction has no potential energy

A box of mass 4 kg is pushed 3 m across a rough level floor and then pushed back to where it started. The coefficient of friction is 0.25. Find the work done by friction on the way out, on the way back, and over the round trip. Compare with the work of gravity on a ball thrown up 3 m and caught again.

Solution

Friction always points against the motion, so it does negative work on both legs:

$$
\begin{aligned}
f &= kmg \\
&= 0.25 \times 4 \times 10 \\
&= 10\,\mathrm{N} \\
W_1 &= -fd \\
&= -10 \times 3 \\
&= -30\,\mathrm{J} \\
W_2 &= -fd \\
&= -10 \times 3 \\
&= -30\,\mathrm{J} \\
W &= W_1 + W_2 \\
&= -30 + (-30) \\
&= -60\,\mathrm{J}
\end{aligned}
$$

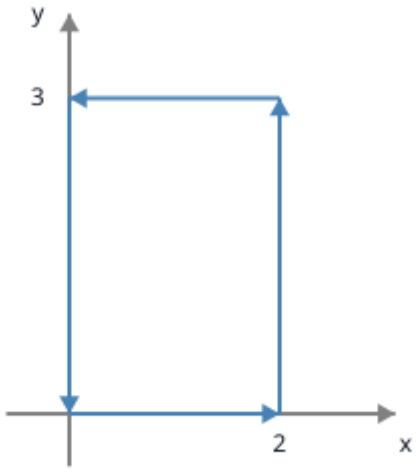
Gravity keeps its direction. It does negative work on the way up and the same amount of positive work on the way down:

$$
\begin{aligned}
G_1 &= -mgd \\
&= -4 \times 10 \times 3 \\
&= -120\,\text{J} \\
G_2 &= mgd \\
&= 4 \times 10 \times 3 \\
&= 120\,\text{J} \\
G &= G_1 + G_2 \\
&= -120 + 120 \\
&= 0\,\text{J}
\end{aligned}
$$

> Result.
>
> Friction does -30 J out, -30 J back and -60 J over the round trip. Gravity does -120 J up, 120 J down and 0 J over its round trip. A force is conservative only if its work round every closed path is zero. Friction fails this test, so no potential energy can be defined for it: the energy it takes becomes heat and cannot be got back by returning.

Question 10. Round a closed path

A particle is taken once round the rectangle with corners (0, 0), $(2, 0)$, $(2, 3)$ and (0, 3), in that order, with coordinates in metres. Find the work done on it over the whole circuit (a) by the force $F_1 = (2xy\,\hat{i} + x^2\,\hat{j})$ N and (b) by the force $F_2 = (-y\,\hat{i} + x\,\hat{j})$ N.



Solution

Add the work leg by leg. On the two legs along x, $dy = 0$; on the two legs along y, $dx = 0$.

(a) For the first force $dW = 2xy\,dx + x^2\,dy$. Along the bottom $y = 0$, up the right side x = 2, back along the top y = 3, down the left side $x = 0$:

$$
\begin{aligned}
A_1 &= 0a^2 \\
&= 0 \times 2^2 \\
&= 0\,\mathrm{J} \\
A_2 &= a^2 b \\
&= 2^2 \times 3 \\
&= 12\,\mathrm{J} \\
A_3 &= -ba^2 \\
&= -3 \times 2^2 \\
&= -12\,\mathrm{J} \\
A_4 &= 0b \\
&= 0 \times 3 \\
&= 0\,\mathrm{J} \\
W_1 &= A_1 + A_2 + A_3 + A_4 \\
&= 0 + 12 + (-12) + 0 \\
&= 0\,\mathrm{J}
\end{aligned}
$$

(b) For the second force $dW = -y\,dx + x\,dy$, on the same four legs:

$$
\begin{aligned}
B_1 &= 0a \\
&= 0 \times 2 \\
&= 0\,\mathrm{J} \\
B_2 &= ab \\
&= 2 \times 3 \\
&= 6\,\mathrm{J} \\
B_3 &= ba \\
&= 3 \times 2 \\
&= 6\,\mathrm{J} \\
B_4 &= 0b \\
&= 0 \times 3 \\
&= 0\,\mathrm{J} \\
W_2 &= B_1 + B_2 + B_3 + B_4 \\
&= 0 + 6 + 6 + 0 \\
&= 12\,\mathrm{J}
\end{aligned}
$$

> Result.
>
> The first force does 0 J round the circuit: it is conservative. The second does 12 J, which is twice the area of the rectangle: it is not conservative, and going round the other way would give -12 J.

Question 11. A weight on a spring: two potential energies together

A block of mass 2 kg hangs from a spring of stiffness 100 N/m. Let x be the stretch of the spring, measured downward from its unstretched length. Write the total potential energy as a function of x, taking both parts as zero at $x = 0$. Find the stretch at which the block can hang at rest, and the potential energy there.

Solution

When the block is lower by x, the spring has gained $\frac{1}{2}kx^2$ and gravity has lost $m\,g\,x$:

$$U(x) = \tfrac{1}{2}\,k\,x^2 - m\,g\,x$$

The net force is minus the slope, $F = -k\,x + m\,g$. The block can rest where this is zero, that is where the potential-energy curve is at its lowest:

$$\begin{aligned} x_0 &= \frac{mg}{k} \\ &= \frac{2 \times 10}{100} \\ &= 0.2\ \text{m} \\ U_0 &= \frac{kx_0^2}{2} - mgx_0 \\ &= \frac{100 \times 0.2^2}{2} - 2 \times 10 \times 0.2 \\ &= -2\ \text{J} \end{aligned}$$

> Result.
>
> The block hangs at rest with the spring stretched by 0.2 m, where the total potential energy has its least value, -2 J. Measured from this point the potential energy is simply $\frac{1}{2}k$ times the square of the displacement: gravity only shifts the centre of the oscillation.

Question 12. A force that depends on distance from a point

A particle is acted on by a force that always points away from a fixed point O and depends only on the distance r from O: $F = c\,r^2$, where c is a constant. Show that it is conservative, and find its potential energy, taking it as zero at O. Find the work it does when the particle goes from the distance r_A to the distance r_B by any path.

Solution

Split any small step into a part along the line from O and a part at right angles to it. The force is along the line from O, so only the first part, dr, counts:

$$dW = F\,dr = c\,r^2\,dr$$

The work therefore depends only on how r changes, not on the path: every force that points to or from a fixed point and depends only on the distance is conservative. Integrating,

Result.

$$W = \tfrac{1}{3}\,c\left(r_B^3 - r_A^3\right)$$

$$U(r) = -\tfrac{1}{3}\,c\,r^3$$

Question 13. Work from a change in potential energy

A particle moves along the x axis where its potential energy is $U = (x^3 - 6x)$ J, with x in metres. Find the work done by the corresponding force when the particle moves from x = 1 m to x = 3 m. If no other force acts, what is the change in its kinetic energy?

Solution

The work done by a conservative force is the fall in its potential energy, from the starting point A to the end point B:

$$W = U_A - U_B$$

$$\begin{aligned} U_1 &= x_1^3 - 6x_1 \\ &= 1^3 - 6 \times 1 \\ &= -5\ \text{J} \\ U_2 &= x_2^3 - 6x_2 \\ &= 3^3 - 6 \times 3 \\ &= 9\ \text{J} \\ W &= U_1 - U_2 \\ &= -5 - 9 \\ &= -14\ \text{J} \end{aligned}$$

Result.

The potential energy rises from -5 J to 9 J, so the force does -14 J of work. If it is the only force, the kinetic energy changes by the same amount, -14 J: the particle must have had at least that much kinetic energy at the start to get there.